Irish Interdisciplinary Journal of Science & Research (IIJSR)
Volume 10, Issue 1, Pages 86-104, January-March 2026

Block Hybrid Method for Solving Linear and Nonlinear Volterra Integro-Differential Equations of the Second-Kind

Ayinde Muhammed Abdullahi[1] & Sabo John[2*]

[1]*Department of Mathematics, University of Abuja, FCT, Nigeria.* [2]*Department of Mathematics, Adamawa State University, Mubi, Nigeria. Corresponding Author (Sabo John) Email: sabojohn21@gmail.com*[*]

DOI: https://doi.org/10.46759/iijsr/2026.10111

Article Received: 17 January 2026 *Article Accepted: 21 March 2026* *Article Published: 28 March 2026*

ABSTRACT

This study presents the development of a new third derivative block hybrid method for solving linear and nonlinear Volterra Integro-Differential Equations (VIDEs) of the second-kind, which frequently arise in modeling physical and engineering processes with memory effects. The new third derivative block hybrid method is constructed using the framework of general linear multistep method and formulated as a single-step block hybrid scheme through Linear Block Algorithm (LBA). By incorporating higher-order derivatives, continuous polynomial approximations are obtained for both the solution and its derivatives, allowing efficient and accurate treatment of the coupled differential and integral terms. The unknown parameters of the method are systematically determined to ensure high-order accuracy and computational efficiency. A comprehensive theoretical analysis of the new LBA is carried out, the results show that the method is of uniform order ten, satisfies the consistency conditions, and is zero-stable. Consequently, convergence is guaranteed. The region of absolute stability is derived using the Boundary Locus Method and illustrated numerically, confirming the robustness of the algorithm for practical applications. To validate the performance of the new LBA scheme, several linear and nonlinear test problems involving second-kind Volterra integro-differential equations are solved. Numerical results are presented in tabular and graphical forms; the comparisons demonstrate that the new LBA produces highly accurate approximations with significantly reduced error across different step sizes.

Keywords: Volterra Integro-Differential Equations; Linear Block Algorithm; Third-Derivative Method; Block Hybrid Method; Boundary Locus Method; Numerical Simulation; Consistency; Zero-Stable; Consequent; Convergence; Block Hybrid Method.

1. Introduction

In engineering and the physical sciences, many natural and physical processes are described using mathematical models formulated as differential or integral equations. These equations appear in various forms in the literature, including stiff, oscillatory, and fractional differential equations, among others (Aloko, 2018). Owing to the complexity of such models, a wide range of solution strategies has been developed, with researchers employing both semi-analytical and purely numerical techniques to obtain accurate and reliable solutions. Integral equations are commonly used to solve various problems in mathematical physics (Matinfar and Riahifar, 2015; Rani and Mishra, 2019). A standard representation of such an equation is given by the form

$$\rho(\xi)=\vartheta(\xi)+\varphi\int_{\omega(\xi)}^{\varpi(\xi)}\mathrm{K}(\xi,\tau)\rho(\tau)d\tau \qquad (1.1)$$

Where φ is a constant parameter, $\mathrm{K}(\xi,\tau)$ is called the kernel of integral equation, $\vartheta(\xi)$ is a function and $\omega(\xi)$ and $\varpi(\xi)$ are the limits of integration can be constants, variables, or a combination of both (Akbar et al. 2019). This work focuses on a class of integro-differential equations of Volterra type, which are characterized by their hereditary structure and are expressed in the form presented in the referenced equation (1.2) as

$$\rho^{(u)}(\xi)=\vartheta(\xi)+\varphi\int_{\kappa}^{\xi}\mathrm{K}(\xi,\tau)\rho(\tau)d\tau \qquad (1.2)$$

Volterra Integro-Differential Equations (VIDEs) have traditionally been addressed using methods such as series solutions, successive approximations, and Laplace transforms (Issa and Hamoud, 2020). However, these approaches often face limitations, including intensive computational requirements, challenges in achieving high-order convergence and the production of series solutions that may be impractical or unrealistic for modeling real-world physical problems (Falade, and Tiamiyu, 2020; Rabiel et al. 2019).

Early contributions combined analytical and numerical perspectives to improve the solution of Volterra integro-differential equations. Okai et al. (2019) transformed linear Volterra integro-differential equations into equivalent second-kind Volterra integral equations and applied a Modified Adomian Decomposition Method, achieving rapid convergence and highly accurate series solutions. In a numerical context, Majid and Mohamed (2019) developed a fifth-order multistep block method using Boole’s quadrature rule, which efficiently handled both linear and nonlinear problems by computing multiple solution points simultaneously, demonstrating strong stability and reduced computational cost. These studies established a foundation for accurate and efficient treatment of Volterra-type equations using both analytical decomposition and block-based numerical schemes.

Subsequent developments emphasized stability-preserving and hybrid solution strategies. Ishak and Selamat (2020) introduced an extended trapezoidal method within a PECE framework that maintained A-stability while delivering higher-order accuracy and improved convergence for first-order linear Volterra integro-differential equations. Similarly, Ahmed (2020) applied the Laplace-Adomian Decomposition Method and the Laplace Iterative Method to systems of Volterra integro-differential equations, demonstrating through linear and nonlinear examples that both approaches yield reliable and accurate approximations closely matching exact solutions. Together, these methods highlight the effectiveness of combining classical numerical schemes with transform- and decomposition-based techniques.

More recent research has focused on high-order, efficient and problem-adapted numerical methods. Olowe et al. (2023) developed a seventh-order trigonometrically fitted block Simpson’s method with strong stability properties, significantly outperforming traditional schemes such as Runge-Kutta-Fehlberg and Adams-Bashforth-Moulton methods. Complementary approaches include fourth-degree hat functions for Volterra integral equations (Mohammed and Khudair, 2023), polynomial collocation techniques for integro-differential equations (Ajileye and Amoo, 2023) and enhanced Adomian-based methods incorporating Laplace transforms and Padé approximants (Anakira et al., 2023). Additionally, Olowe et al. (2024) developed a fifth-order trigonometrically fitted block method that showed superior accuracy and efficiency, particularly for oscillatory problems. Collectively, these works reflect sustained progress toward stable, high-accuracy, and computationally efficient numerical frameworks for solving complex Volterra integral and integro-differential equations in applied science and engineering.

1.1. Aim and Objectives of the study

The aim of this research is to develop a block hybrid method for solving linear and nonlinear volterra integro-differential equations of the second-kind. The following are objectives:

1) To derive some third derivative numerical scheme using the LBA,
2) To obtain a continuous form of third derivative numerical scheme and its higher order numerical schemes,

3) To established the basic properties of the numerical scheme,

4) To numerically integrate the Block Hybrid Method (BHM) on linear and nonlinear second-kind Volterra integro-differential equations and present a comparison in both tabular and textual forms.

2. Material and Method

This section describes the development of Block Hybrid Method (BHM) through third derivative linear block algorithm for numerical integration of Volterra integro-differential equation of second kind of the form (1.2). The following proposition 2.1 was used to derive the BHM using the Linear Block Algorithm (LBA) following the methods of Adeyeye and Omar, (2019) with the help of Scientific Workplace 5.5 Software Package.

We adopt the general linear multistep method of the form

$$\sum_{j=0}^{1}\alpha_j \rho_{n+j} = h^{\mu}\sum_{j=0}^{1}\beta_j \vartheta_{n+j} \qquad (2.1)$$

2.1. Derivation of Block Hybrid Method (BHM)

Proposition 2.1

Consider the general linear multistep method (2.1) with single step block hybrid method, which exist one numerical scheme with the linear block algorithm of the form

$$\rho_{n+\eta} = \sum_{j=0}^{2}\frac{(\eta h)^j}{j!}\rho_n^{(j)} + \sum_{j=0}^{1}\left(\Lambda_{j\eta}\vartheta_{n+j}\right), \quad \eta = -\frac{1}{7}, -\frac{2}{7}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1 \qquad (2.2)$$

together with its higher derivatives

$$\rho_{n+\eta}^{\sigma} = \sum_{j=0}^{2-\tau}\frac{(\eta h)^j}{j!}\rho_n^{(j+\sigma)} + \sum_{j=0}^{7}\left(\mathrm{X}_{j\sigma}\,\vartheta_{n+j}\right), \sigma = 1_{\left(-\frac{1}{7}, -\frac{2}{7}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right)}, \sigma = 2_{\left(-\frac{1}{7}, -\frac{2}{7}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right)} \qquad (2.3)$$

is consider, with $\Lambda_{\eta j} = \Psi^{-1}\mathrm{Z}$ and $\mathrm{X}_{\eta j\sigma} = \Psi^{-1}\mathrm{E}$ were

$$\Psi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{\left(-\frac{1}{7}h\right)^{1}}{1!} & \frac{\left(-\frac{2}{7}h\right)^{1}}{1!} & \frac{(0)^{1}}{1!} & \frac{\left(\frac{1}{7}h\right)^{1}}{1!} & \frac{\left(\frac{2}{7}h\right)^{1}}{1!} & \frac{\left(\frac{3}{7}h\right)^{1}}{1!} & \frac{\left(\frac{4}{7}h\right)^{1}}{1!} & \frac{\left(\frac{5}{7}h\right)^{1}}{1!} & \frac{\left(\frac{6}{7}h\right)^{1}}{1!} & \frac{(h)^{1}}{1!} \\
\frac{\left(-\frac{1}{7}h\right)^{2}}{2!} & \frac{\left(-\frac{2}{7}h\right)^{2}}{2!} & \frac{(0)^{2}}{2!} & \frac{\left(\frac{1}{7}h\right)^{2}}{2!} & \frac{\left(\frac{2}{7}h\right)^{2}}{2!} & \frac{\left(\frac{3}{7}h\right)^{2}}{2!} & \frac{\left(\frac{4}{7}h\right)^{2}}{2!} & \frac{\left(\frac{5}{7}h\right)^{2}}{2!} & \frac{\left(\frac{6}{7}h\right)^{2}}{2!} & \frac{(h)^{2}}{2!} \\
\frac{\left(-\frac{1}{7}h\right)^{3}}{3!} & \frac{\left(-\frac{2}{7}h\right)^{3}}{3!} & \frac{(0)^{3}}{3!} & \frac{\left(\frac{1}{7}h\right)^{3}}{3!} & \frac{\left(\frac{2}{7}h\right)^{3}}{3!} & \frac{\left(\frac{3}{7}h\right)^{3}}{3!} & \frac{\left(\frac{4}{7}h\right)^{3}}{3!} & \frac{\left(\frac{5}{7}h\right)^{3}}{3!} & \frac{\left(\frac{6}{7}h\right)^{3}}{3!} & \frac{(h)^{3}}{3!} \\
\frac{\left(-\frac{1}{7}h\right)^{4}}{4!} & \frac{\left(-\frac{2}{7}h\right)^{4}}{4!} & \frac{(0)^{4}}{4!} & \frac{\left(\frac{1}{7}h\right)^{4}}{4!} & \frac{\left(\frac{2}{7}h\right)^{4}}{4!} & \frac{\left(\frac{3}{7}h\right)^{4}}{4!} & \frac{\left(\frac{4}{7}h\right)^{4}}{4!} & \frac{\left(\frac{5}{7}h\right)^{4}}{4!} & \frac{\left(\frac{6}{7}h\right)^{4}}{4!} & \frac{(h)^{4}}{4!} \\
\frac{\left(-\frac{1}{7}h\right)^{5}}{5!} & \frac{\left(-\frac{2}{7}h\right)^{5}}{5!} & \frac{(0)^{5}}{5!} & \frac{\left(\frac{1}{7}h\right)^{5}}{5!} & \frac{\left(\frac{2}{7}h\right)^{5}}{5!} & \frac{\left(\frac{3}{7}h\right)^{5}}{5!} & \frac{\left(\frac{4}{7}h\right)^{5}}{5!} & \frac{\left(\frac{5}{7}h\right)^{5}}{5!} & \frac{\left(\frac{6}{7}h\right)^{5}}{5!} & \frac{(h)^{5}}{5!} \\
\frac{\left(-\frac{1}{7}h\right)^{6}}{6!} & \frac{\left(-\frac{2}{7}h\right)^{6}}{6!} & \frac{(0)^{6}}{6!} & \frac{\left(\frac{1}{7}h\right)^{6}}{6!} & \frac{\left(\frac{2}{7}h\right)^{6}}{6!} & \frac{\left(\frac{3}{7}h\right)^{6}}{6!} & \frac{\left(\frac{4}{7}h\right)^{6}}{6!} & \frac{\left(\frac{5}{7}h\right)^{6}}{6!} & \frac{\left(\frac{6}{7}h\right)^{6}}{6!} & \frac{(h)^{6}}{6!} \\
\frac{\left(-\frac{1}{7}h\right)^{7}}{7!} & \frac{\left(-\frac{2}{7}h\right)^{7}}{7!} & \frac{(0)^{7}}{7!} & \frac{\left(\frac{1}{7}h\right)^{7}}{7!} & \frac{\left(\frac{2}{7}h\right)^{7}}{7!} & \frac{\left(\frac{3}{7}h\right)^{7}}{7!} & \frac{\left(\frac{4}{7}h\right)^{7}}{7!} & \frac{\left(\frac{5}{7}h\right)^{7}}{7!} & \frac{\left(\frac{6}{7}h\right)^{7}}{7!} & \frac{(h)^{7}}{7!} \\
\frac{\left(-\frac{1}{7}h\right)^{8}}{8!} & \frac{\left(-\frac{2}{7}h\right)^{8}}{8!} & \frac{(0)^{8}}{8!} & \frac{\left(\frac{1}{7}h\right)^{8}}{8!} & \frac{\left(\frac{2}{7}h\right)^{8}}{8!} & \frac{\left(\frac{3}{7}h\right)^{8}}{8!} & \frac{\left(\frac{4}{7}h\right)^{8}}{8!} & \frac{\left(\frac{5}{7}h\right)^{8}}{8!} & \frac{\left(\frac{6}{7}h\right)^{8}}{8!} & \frac{(h)^{8}}{8!} \\
\frac{\left(-\frac{1}{7}h\right)^{9}}{9!} & \frac{\left(-\frac{2}{7}h\right)^{9}}{9!} & \frac{(0)^{9}}{9!} & \frac{\left(\frac{1}{7}h\right)^{9}}{9!} & \frac{\left(\frac{2}{7}h\right)^{9}}{9!} & \frac{\left(\frac{3}{7}h\right)^{9}}{9!} & \frac{\left(\frac{4}{7}h\right)^{9}}{9!} & \frac{\left(\frac{5}{7}h\right)^{9}}{9!} & \frac{\left(\frac{6}{7}h\right)^{9}}{9!} & \frac{(h)^{9}}{9!}
\end{pmatrix}, \mathrm{Z} = \begin{pmatrix}
\frac{(\eta h)^{3}}{3!} \\ \frac{(\eta h)^{4}}{4!} \\ \frac{(\eta h)^{5}}{5!} \\ \frac{(\eta h)^{6}}{6!} \\ \frac{(\eta h)^{7}}{7!} \\ \frac{(\eta h)^{8}}{8!} \\ \frac{(\eta h)^{9}}{9!} \\ \frac{(\eta h)^{10}}{10!} \\ \frac{(\eta h)^{11}}{11!} \\ \frac{(\eta h)^{12}}{12!}
\end{pmatrix}, \mathrm{E} = \begin{pmatrix}
\frac{(\eta h)^{3-\sigma}}{(3-\sigma)!} \\ \frac{(\eta h)^{4-\sigma}}{(4-\sigma)!} \\ \frac{(\eta h)^{5-\sigma}}{(5-\sigma)!} \\ \frac{(\eta h)^{6-\sigma}}{(6-\sigma)!} \\ \frac{(\eta h)^{7-\sigma}}{(7-\sigma)!} \\ \frac{(\eta h)^{8-\sigma}}{(8-\sigma)!} \\ \frac{(\eta h)^{9-\sigma}}{(9-\sigma)!} \\ \frac{(\eta h)^{10-\sigma}}{(10-\sigma)!} \\ \frac{(\eta h)^{11-\sigma}}{(11-\sigma)!} \\ \frac{(\eta h)^{12-\sigma}}{(12-\sigma)!}
\end{pmatrix}$$

Proof

Solving equation (2.2) and (2.3) one by one to yield the continuous schemes in form of polynomial as

$$\begin{aligned}\rho(\xi_n + \eta h) = {} & \alpha_{\frac{1}{7}}\rho_{n+\frac{1}{7}} + \alpha_{\frac{2}{7}}\rho_{n+\frac{2}{7}} + \alpha_{\frac{3}{7}}\rho_{n+\frac{3}{7}} + \\ & h^3\left(\beta_{-\frac{1}{7}}\vartheta_{n-\frac{1}{7}} + \beta_{-\frac{2}{7}}\vartheta_{n-\frac{2}{7}} + \beta_0\vartheta_n + \beta_{\frac{1}{7}}\vartheta_{n+\frac{1}{7}} + \beta_{\frac{2}{7}}\vartheta_{n+\frac{2}{7}} + \beta_{\frac{3}{7}}\vartheta_{n+\frac{3}{7}} + \beta_{\frac{4}{7}}\vartheta_{n+\frac{4}{7}} + \beta_{\frac{5}{7}}\vartheta_{n+\frac{5}{7}} + \beta_{\frac{6}{7}}\vartheta_{n+\frac{6}{7}} + \beta_1\vartheta_{n+1}\right)\end{aligned} \qquad (2.4)$$

Where $\eta = \xi_n + \xi h$ in the equation (2.4) and

$$\left.\begin{aligned}
&\alpha_{\frac{1}{7}} = 3-\frac{35}{2}\eta+\frac{49}{2}\eta^2,\ \alpha_{\frac{2}{7}} = -3+28\eta-49\eta^2,\ \alpha_{\frac{1}{7}} = 1-\frac{21}{2}\eta+\frac{49}{2}\eta^2,\\
&\beta_{-\frac{1}{7}} = -\frac{803}{165957120}-\frac{30571}{558835200}\eta+\frac{2734121}{2235340800}\eta^2-\frac{7}{96}\eta^4+\frac{2051}{4800}\eta^5-\frac{161651}{172800}\eta^6+\frac{9947}{86400}\eta^7-\frac{977207}{276480}\eta^8+\frac{789929}{103680}\eta^9\\
&-\frac{15647317}{2073600}\eta^{10}+\frac{10706059}{2851200}\eta^{11}-\frac{5764801}{7603200}\eta^{12}\\
&\beta_{-\frac{1}{3}} = \frac{373}{829785600}+\frac{37}{17463600}\eta-\frac{499519}{6706022400}\eta^2+\frac{7}{1728}\eta^4-\frac{1561}{86400}\eta^5+\frac{49}{97200}\eta^6-\frac{34643}{2800}\eta^7+\frac{549829}{829440}\eta^8-\frac{218491}{207360}\eta^9\\
&+\frac{5764801}{6220800}\eta^{10}-\frac{823543}{1900800}\eta^{11}+\frac{5764801}{68428800}\eta^{12}\\
&\beta_0 = -\frac{11}{29635200}+\frac{160309}{97796160}\eta-\frac{1048601}{37255680}\eta^2+\frac{1}{6}\eta^3-\frac{15}{160}\eta^4+\frac{553}{864}\eta^5-\frac{62671}{17280}\eta^6+\frac{10633}{2880}\eta^7-\frac{175273}{23040}\eta^8+\frac{84035}{3456}\eta^9\\
&-\frac{117649}{4320}\eta^{10}+\frac{823543}{57024}\eta^{11}-\frac{5764801}{1900800}\eta^{12}\\
&\beta_{\frac{1}{7}} = -\frac{292309}{207446400}+\frac{5039777}{244490400}\eta-\frac{48525139}{558835200}\eta^2+\frac{49}{72}\eta^4-\frac{637}{3600}\eta^5-\frac{757687}{129600}\eta^6+\frac{226037}{21600}\eta^7-\frac{559433}{69120}\eta^8+\frac{2336173}{15184011}\eta^9\\
&-\frac{823543}{14400}\eta^{10}+\frac{5764801}{178200}\eta^{11}-\frac{40353607}{5702400}\eta^{12}\\
&\beta_{\frac{2}{7}} = -\frac{651377}{414892800}+\frac{539591}{34927200}\eta-\frac{27575231}{1117670400}\eta^2-\frac{49}{96}\eta^4+\frac{4067}{4800}\eta^5+\frac{466823}{86400}\eta^6-\frac{1259839}{86400}\eta^7+\frac{501809}{138240}\eta^8-\frac{5563117}{103680}\eta^9\\
&+\frac{79883671}{1036800}\eta^{10}-\frac{132590423}{2851200}\eta^{11}+\frac{40353607}{3801600}\eta^{12}\\
&\beta_{\frac{3}{7}} = \frac{47531}{414892800}-\frac{80039}{177811200}\eta-\frac{116047}{17740800}\eta^2+\frac{49}{144}\eta^4-\frac{15631}{21600}\eta^5+\frac{298753}{86400}\eta^6-\frac{177331}{14400}\eta^7+\frac{40817}{46080}\eta^8-\frac{184877}{4320}\eta^9\\
&+\frac{23882747}{345600}\eta^{10}-\frac{5764801}{129600}\eta^{11}+\frac{40353607}{3801600}\eta^{12}\\
&\beta_{\frac{4}{7}} = -\frac{517}{8297856}+\frac{138529}{488980800}\eta+\frac{567667}{186278400}\eta^2-\frac{49}{288}\eta^4+\frac{2891}{7200}\eta^5+\frac{429779}{259200}\eta^6-\frac{293951}{43200}\eta^7+\frac{45619}{23040}\eta^8+\frac{1193297}{51840}\eta^9\\
&-\frac{10706059}{259200}\eta^{10}+\frac{40353607}{1425600}\eta^{11}-\frac{40353607}{5702400}\eta^{12}\\
&\beta_{\frac{5}{7}} = \frac{83}{4233600}-\frac{1123}{13970880}\eta-\frac{118931}{111767040}\eta^2+\frac{7}{120}\eta^4-\frac{7}{48}\eta^5-\frac{4753}{8640}\eta^6+\frac{10633}{4320}\eta^7-\frac{74431}{69120}\eta^8-\frac{84035}{10368}\eta^9+\frac{823543}{51840}\eta^{10}\\
&-\frac{823543}{71280}\eta^{11}+\frac{5764801}{1900800}\eta^{12}\\
&\beta_{\frac{6}{7}} = -\frac{3179}{829785600}+\frac{877}{61122600}\eta+\frac{167257}{745113600}\eta^2-\frac{7}{576}\eta^4+\frac{2723}{86400}\eta^5+\frac{9653}{86400}\eta^6-\frac{30527}{57600}\eta^7+\frac{26411}{92160}\eta^8+\frac{117649}{69120}\eta^9-\\
&\frac{823543}{230400}\eta^{10}+\frac{15647317}{5702400}\eta^{11}-\frac{5764801}{7603200}\eta^{12}\\
&\beta_1 = \frac{289}{829785600}-\frac{1579}{1303948800}\eta-\frac{144509}{6706022400}\eta^2+\frac{1}{864}\eta^4-\frac{133}{43200}\eta^5-\frac{16219}{1555200}\eta^6+\frac{4459}{86400}\eta^7-\frac{26411}{829440}\eta^8-\frac{16807}{103680}\eta^9\\
&+\frac{2235331}{6220800}\eta^{10}-\frac{823543}{2851200}\eta^{11}+\frac{5764801}{68428800}\eta^{12}
\end{aligned}\right\} \quad (2.5)$$

Expanding the generalized algorithm (2.2) to give a new numerical scheme as

$$\left.\begin{aligned}
&\rho_{n-\frac{1}{7}} = \rho_n - \frac{1}{7}h\rho'_n + \frac{\left(-\frac{1}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{7}}+\Lambda_{11}\vartheta_{n-\frac{2}{7}}+\Lambda_{12}\vartheta_n+\Lambda_{13}\vartheta_{n+\frac{1}{7}}+\Lambda_{14}\vartheta_{n+\frac{2}{7}}+\Lambda_{15}\vartheta_{n+\frac{3}{7}}+\Lambda_{16}\vartheta_{n+\frac{4}{7}}+\Lambda_{17}\vartheta_{n+\frac{5}{7}}+\Lambda_{18}\vartheta_{n+\frac{6}{7}}+\Lambda_{19}\vartheta_{n+1}\right)\\
&\rho_{n-\frac{2}{7}} = \rho_n - \frac{2}{7}h\rho'_n + \frac{\left(-\frac{2}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{7}}+\Lambda_{21}\vartheta_{n-\frac{2}{7}}+\Lambda_{22}\vartheta_n+\Lambda_{23}\vartheta_{n+\frac{1}{7}}+\Lambda_{24}\vartheta_{n+\frac{2}{7}}+\Lambda_{25}\vartheta_{n+\frac{3}{7}}+\Lambda_{26}\vartheta_{n+\frac{4}{7}}+\Lambda_{27}\vartheta_{n+\frac{5}{7}}+\Lambda_{28}\vartheta_{n+\frac{6}{7}}+\Lambda_{29}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{1}{7}} = \rho_n + \frac{1}{7}h\rho'_n + \frac{\left(\frac{1}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{7}}+\Lambda_{31}\vartheta_{n-\frac{2}{7}}+\Lambda_{32}\vartheta_n+\Lambda_{33}\vartheta_{n+\frac{1}{7}}+\Lambda_{34}\vartheta_{n+\frac{2}{7}}+\Lambda_{35}\vartheta_{n+\frac{3}{7}}+\Lambda_{36}\vartheta_{n+\frac{4}{7}}+\Lambda_{37}\vartheta_{n+\frac{5}{7}}+\Lambda_{38}\vartheta_{n+\frac{6}{7}}+\Lambda_{39}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{2}{7}} = \rho_n + \frac{2}{7}h\rho'_n + \frac{\left(\frac{2}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{7}}+\Lambda_{41}\vartheta_{n-\frac{2}{7}}+\Lambda_{42}\vartheta_n+\Lambda_{43}\vartheta_{n+\frac{1}{7}}+\Lambda_{44}\vartheta_{n+\frac{2}{7}}+\Lambda_{45}\vartheta_{n+\frac{3}{7}}+\Lambda_{46}\vartheta_{n+\frac{4}{7}}+\Lambda_{47}\vartheta_{n+\frac{5}{7}}+\Lambda_{48}\vartheta_{n+\frac{6}{7}}+\Lambda_{49}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{3}{7}} = \rho_n + \frac{3}{7}h\rho'_n + \frac{\left(\frac{3}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{7}}+\Lambda_{51}\vartheta_{n-\frac{2}{7}}+\Lambda_{52}\vartheta_n+\Lambda_{53}\vartheta_{n+\frac{1}{7}}+\Lambda_{54}\vartheta_{n+\frac{2}{7}}+\Lambda_{55}\vartheta_{n+\frac{3}{7}}+\Lambda_{56}\vartheta_{n+\frac{4}{7}}+\Lambda_{57}\vartheta_{n+\frac{5}{7}}+\Lambda_{58}\vartheta_{n+\frac{6}{7}}+\Lambda_{59}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{4}{7}} = \rho_n + \frac{4}{7}h\rho'_n + \frac{\left(\frac{4}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{7}}+\Lambda_{61}\vartheta_{n-\frac{2}{7}}+\Lambda_{62}\vartheta_n+\Lambda_{63}\vartheta_{n+\frac{1}{7}}+\Lambda_{64}\vartheta_{n+\frac{2}{7}}+\Lambda_{65}\vartheta_{n+\frac{3}{7}}+\Lambda_{66}\vartheta_{n+\frac{4}{7}}+\Lambda_{67}\vartheta_{n+\frac{5}{7}}+\Lambda_{68}\vartheta_{n+\frac{6}{7}}+\Lambda_{69}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{5}{7}} = \rho_n + \frac{5}{7}h\rho'_n + \frac{\left(\frac{5}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{70}\vartheta_{n-\frac{1}{7}}+\Lambda_{71}\vartheta_{n-\frac{2}{7}}+\Lambda_{72}\vartheta_n+\Lambda_{73}\vartheta_{n+\frac{1}{7}}+\Lambda_{74}\vartheta_{n+\frac{2}{7}}+\Lambda_{75}\vartheta_{n+\frac{3}{7}}+\Lambda_{76}\vartheta_{n+\frac{4}{7}}+\Lambda_{77}\vartheta_{n+\frac{5}{7}}+\Lambda_{78}\vartheta_{n+\frac{6}{7}}+\Lambda_{79}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{6}{7}} = \rho_n + \frac{6}{7}h\rho'_n + \frac{\left(\frac{6}{7}h\right)^2}{2!}\rho''_n + h^3\left(\Lambda_{80}\vartheta_{n-\frac{1}{7}}+\Lambda_{81}\vartheta_{n-\frac{2}{7}}+\Lambda_{82}\vartheta_n+\Lambda_{83}\vartheta_{n+\frac{1}{7}}+\Lambda_{84}\vartheta_{n+\frac{2}{7}}+\Lambda_{85}\vartheta_{n+\frac{3}{7}}+\Lambda_{86}\vartheta_{n+\frac{4}{7}}+\Lambda_{87}\vartheta_{n+\frac{5}{7}}+\Lambda_{88}\vartheta_{n+\frac{6}{7}}+\Lambda_{89}\vartheta_{n+1}\right)\\
&\rho_{n+1} = \rho_n + h\rho'_n + \frac{(h)^2}{2!}\rho''_n + h^3\left(\Lambda_{90}\vartheta_{n-\frac{1}{7}}+\Lambda_{91}\vartheta_{n-\frac{2}{7}}+\Lambda_{92}\vartheta_n+\Lambda_{93}\vartheta_{n+\frac{1}{7}}+\Lambda_{94}\vartheta_{n+\frac{2}{7}}+\Lambda_{95}\vartheta_{n+\frac{3}{7}}+\Lambda_{96}\vartheta_{n+\frac{4}{7}}+\Lambda_{97}\vartheta_{n+\frac{5}{7}}+\Lambda_{98}\vartheta_{n+\frac{6}{7}}+\Lambda_{99}\vartheta_{n+1}\right)
\end{aligned}\right\} \quad (2.6)$$

Also, expand the higher derivative of the generalized (2.3)

$$\left.\begin{aligned}
\rho'_{n-\frac{1}{7}} &= \rho'_n - \frac{1}{7}h\rho''_n + h^2 f\left(X_{110}\vartheta_{n-\frac{1}{7}} + X_{111}\vartheta_{n-\frac{2}{7}} + X_{112}\vartheta_n + X_{113}\vartheta_{n+\frac{1}{7}} + X_{114}\vartheta_{n+\frac{2}{7}} + X_{115}\vartheta_{n+\frac{3}{7}} + X_{116}\vartheta_{n+\frac{4}{7}} + X_{117}\vartheta_{n+\frac{5}{7}} + X_{118}\vartheta_{n+\frac{6}{7}} + X_{119}\vartheta_{n+1}\right)\\
\rho'_{n-\frac{2}{7}} &= \rho'_n - \frac{2}{7}h\rho''_n + h^2\left(X_{120}\vartheta_{n-\frac{1}{7}} + X_{121}\vartheta_{n-\frac{2}{7}} + X_{122}\vartheta_n + X_{123}\vartheta_{n+\frac{1}{7}} + X_{124}\vartheta_{n+\frac{2}{7}} + X_{125}\vartheta_{n+\frac{3}{7}} + X_{126}\vartheta_{n+\frac{4}{7}} + X_{127}\vartheta_{n+\frac{5}{7}} + X_{128}\vartheta_{n+\frac{6}{7}} + X_{129}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{1}{7}} &= \rho'_n + \frac{1}{7}h\rho''_n + h^2\left(X_{130}\vartheta_{n-\frac{1}{7}} + X_{131}\vartheta_{n-\frac{2}{7}} + X_{132}\vartheta_n + X_{133}\vartheta_{n+\frac{1}{7}} + X_{134}\vartheta_{n+\frac{2}{7}} + X_{135}\vartheta_{n+\frac{3}{7}} + X_{136}\vartheta_{n+\frac{4}{7}} + X_{137}\vartheta_{n+\frac{5}{7}} + X_{138}\vartheta_{n+\frac{6}{7}} + X_{139}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{2}{7}} &= \rho'_n + \frac{2}{7}h\rho''_n + h^2\left(X_{140}\vartheta_{n-\frac{1}{7}} + X_{141}\vartheta_{n-\frac{2}{7}} + X_{142}\vartheta_n + X_{143}\vartheta_{n+\frac{1}{7}} + X_{144}\vartheta_{n+\frac{2}{7}} + X_{145}\vartheta_{n+\frac{3}{7}} + X_{146}\vartheta_{n+\frac{4}{7}} + X_{147}\vartheta_{n+\frac{5}{7}} + X_{148}\vartheta_{n+\frac{6}{7}} + X_{149}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{3}{7}} &= \rho'_n + \frac{3}{7}h\rho''_n + h^2\left(X_{150}\vartheta_{n-\frac{1}{7}} + X_{151}\vartheta_{n-\frac{2}{7}} + X_{152}\vartheta_n + X_{153}\vartheta_{n+\frac{1}{7}} + X_{154}\vartheta_{n+\frac{2}{7}} + X_{155}\vartheta_{n+\frac{3}{7}} + X_{156}\vartheta_{n+\frac{4}{7}} + X_{157}\vartheta_{n+\frac{5}{7}} + X_{158}\vartheta_{n+\frac{6}{7}} + X_{159}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{4}{7}} &= \rho'_n + \frac{4}{7}h\rho''_n + h^2\left(X_{160}\vartheta_{n-\frac{1}{7}} + X_{161}\vartheta_{n-\frac{2}{7}} + X_{162}\vartheta_n + X_{163}\vartheta_{n+\frac{1}{7}} + X_{164}\vartheta_{n+\frac{2}{7}} + X_{165}\vartheta_{n+\frac{3}{7}} + X_{166}\vartheta_{n+\frac{4}{7}} + X_{167}\vartheta_{n+\frac{5}{7}} + X_{168}\vartheta_{n+\frac{6}{7}} + X_{169}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{5}{7}} &= \rho'_n + \frac{5}{7}h\rho''_n + h^2\left(X_{170}\vartheta_{n-\frac{1}{7}} + X_{171}\vartheta_{n-\frac{2}{7}} + X_{172}\vartheta_n + X_{173}\vartheta_{n+\frac{1}{7}} + X_{174}\vartheta_{n+\frac{2}{7}} + X_{175}\vartheta_{n+\frac{3}{7}} + X_{176}\vartheta_{n+\frac{4}{7}} + X_{177}\vartheta_{n+\frac{5}{7}} + X_{178}\vartheta_{n+\frac{6}{7}} + X_{179}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{6}{7}} &= \rho'_n + \frac{5}{7}h\rho''_n + h^2\left(X_{180}\vartheta_{n-\frac{1}{7}} + X_{181}\vartheta_{n-\frac{2}{7}} + X_{182}\vartheta_n + X_{183}\vartheta_{n+\frac{1}{7}} + X_{184}\vartheta_{n+\frac{2}{7}} + X_{185}\vartheta_{n+\frac{3}{7}} + X_{186}\vartheta_{n+\frac{4}{7}} + X_{187}\vartheta_{n+\frac{5}{7}} + X_{188}\vartheta_{n+\frac{6}{7}} + X_{189}\vartheta_{n+1}\right)\\
\rho'_{n+1} &= \rho'_n + h\rho''_n + h^2\left(X_{190}\vartheta_{n-\frac{1}{7}} + X_{191}\vartheta_{n-\frac{2}{7}} + X_{192}\vartheta_n + X_{193}\vartheta_{n+\frac{1}{7}} + X_{194}\vartheta_{n+\frac{2}{7}} + X_{195}\vartheta_{n+\frac{3}{7}} + X_{196}\vartheta_{n+\frac{4}{7}} + X_{197}\vartheta_{n+\frac{5}{7}} + X_{198}\vartheta_{n+\frac{6}{7}} + X_{199}\vartheta_{n+1}\right)
\end{aligned}\right\} \quad (2.7)$$

$$\left.\begin{aligned}
\rho''_{n-\frac{1}{7}} &= \rho''_n + h\left(X_{210}\vartheta_{n-\frac{1}{7}} + X_{211}\vartheta_{n-\frac{2}{7}} + X_{212}\vartheta_n + X_{213}\vartheta_{n+\frac{1}{7}} + X_{214}\vartheta_{n+\frac{2}{7}} + X_{215}\vartheta_{n+\frac{3}{7}} + X_{216}\vartheta_{n+\frac{4}{7}} + X_{217}\vartheta_{n+\frac{5}{7}} + X_{218}\vartheta_{n+\frac{6}{7}} + X_{219}\vartheta_{n+1}\right)\\
\rho''_{n-\frac{2}{7}} &= \rho''_n + h\left(X_{220}\vartheta_{n-\frac{1}{7}} + X_{221}\vartheta_{n-\frac{2}{7}} + X_{222}\vartheta_n + X_{223}\vartheta_{n+\frac{1}{7}} + X_{224}\vartheta_{n+\frac{2}{7}} + X_{225}\vartheta_{n+\frac{3}{7}} + X_{226}\vartheta_{n+\frac{4}{7}} + X_{227}\vartheta_{n+\frac{5}{7}} + X_{228}\vartheta_{n+\frac{6}{7}} + X_{229}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{1}{7}} &= \rho''_n + h\left(X_{230}\vartheta_{n-\frac{1}{7}} + X_{231}\vartheta_{n-\frac{2}{7}} + X_{232}\vartheta_n + X_{233}\vartheta_{n+\frac{1}{7}} + X_{234}\vartheta_{n+\frac{2}{7}} + X_{235}\vartheta_{n+\frac{3}{7}} + X_{236}\vartheta_{n+\frac{4}{7}} + X_{237}\vartheta_{n+\frac{5}{7}} + X_{238}\vartheta_{n+\frac{6}{7}} + X_{239}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{2}{7}} &= \rho''_n + h\left(X_{240}\vartheta_{n-\frac{1}{7}} + X_{241}\vartheta_{n-\frac{2}{7}} + X_{242}\vartheta_n + X_{243}\vartheta_{n+\frac{1}{7}} + X_{244}\vartheta_{n+\frac{2}{7}} + X_{245}\vartheta_{n+\frac{3}{7}} + X_{246}\vartheta_{n+\frac{4}{7}} + X_{247}\vartheta_{n+\frac{5}{7}} + X_{248}\vartheta_{n+\frac{6}{7}} + X_{249}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{3}{7}} &= \rho''_n + h\left(X_{250}\vartheta_{n-\frac{1}{7}} + X_{251}\vartheta_{n-\frac{2}{7}} + X_{252}\vartheta_n + X_{253}\vartheta_{n+\frac{1}{7}} + X_{254}\vartheta_{n+\frac{2}{7}} + X_{255}\vartheta_{n+\frac{3}{7}} + X_{256}\vartheta_{n+\frac{4}{7}} + X_{257}\vartheta_{n+\frac{5}{7}} + X_{258}\vartheta_{n+\frac{6}{7}} + X_{259}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{4}{7}} &= \rho''_n + h\left(X_{260}\vartheta_{n-\frac{1}{7}} + X_{261}\vartheta_{n-\frac{2}{7}} + X_{262}\vartheta_n + X_{263}\vartheta_{n+\frac{1}{7}} + X_{264}\vartheta_{n+\frac{2}{7}} + X_{265}\vartheta_{n+\frac{3}{7}} + X_{266}\vartheta_{n+\frac{4}{7}} + X_{267}\vartheta_{n+\frac{5}{7}} + X_{268}\vartheta_{n+\frac{6}{7}} + X_{269}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{5}{7}} &= \rho''_n + h\left(X_{270}\vartheta_{n-\frac{1}{7}} + X_{271}\vartheta_{n-\frac{2}{7}} + X_{272}\vartheta_n + X_{273}\vartheta_{n+\frac{1}{7}} + X_{274}\vartheta_{n+\frac{2}{7}} + X_{275}\vartheta_{n+\frac{3}{7}} + X_{276}\vartheta_{n+\frac{4}{7}} + X_{277}\vartheta_{n+\frac{5}{7}} + X_{278}\vartheta_{n+\frac{6}{7}} + X_{279}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{6}{7}} &= \rho''_n + h\left(X_{280}\vartheta_{n-\frac{1}{7}} + X_{281}\vartheta_{n-\frac{2}{7}} + X_{282}\vartheta_n + X_{283}\vartheta_{n+\frac{1}{7}} + X_{284}\vartheta_{n+\frac{2}{7}} + X_{285}\vartheta_{n+\frac{3}{7}} + X_{286}\vartheta_{n+\frac{4}{7}} + X_{287}\vartheta_{n+\frac{5}{7}} + X_{288}\vartheta_{n+\frac{6}{7}} + X_{289}\vartheta_{n+1}\right)\\
\rho''_{n+1} &= \rho''_n + h\left(X_{290}\vartheta_{n-\frac{1}{7}} + X_{291}\vartheta_{n-\frac{2}{7}} + X_{292}\vartheta_n + X_{293}\vartheta_{n+\frac{1}{7}} + X_{294}\vartheta_{n+\frac{2}{7}} + X_{295}\vartheta_{n+\frac{3}{7}} + X_{296}\vartheta_{n+\frac{4}{7}} + X_{297}\vartheta_{n+\frac{5}{7}} + X_{298}\vartheta_{n+\frac{6}{7}} + X_{299}\vartheta_{n+1}\right)
\end{aligned}\right\} \quad (2.8)$$

The unknown coefficient of Λ in (2.6) can be seen in appendix after Simplifying $\Lambda_{\eta j} = \Psi^{-1}Z$

Similarly, the unknown coefficients of the higher derivative X in (2.7) and (2.8) are given in appendix after simplify $\mathrm{X}_{\eta j\sigma} = \Psi^{-1}\mathrm{E}$

3. Analysis of Basic Properties of new Block Hybrid Method (BHM)

The analysis of the basic properties of the new BHM was analyzed. These properties are order and error constant, consistency, zero-stability, convergent and region of absolute stability (Raymond et al. 2023).

3.1. Order and Error Constant of BHM

The corollary 3.1 and corollary 3.2 to obtained the order and error constant of BHM

Corollary 3.1

The linear operator $L[\rho(\xi_n);h]$ associate with the local truncation error of the BHM defined in (2.6) to (2.8) is given as

$$C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{13}),\ C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{12}),\ C_{10}h^{10}\rho^{10}(\xi_n)+0(h^{11}).$$

Proof

Consider the linear difference operators associated with (2.6) to (2.8) are given by

$$\left.\begin{aligned}
L[\rho(\xi_n);h] &= \rho_{n-\frac{1}{7}} - \rho_n + \frac{1}{7}h\rho'_n - \frac{\left(-\frac{1}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{7}} + \Lambda_{11}\vartheta_{n-\frac{2}{7}} + \Lambda_{12}\vartheta_n + \Lambda_{13}\vartheta_{n+\frac{1}{7}} + \Lambda_{14}\vartheta_{n+\frac{2}{7}} + \Lambda_{15}\vartheta_{n+\frac{3}{7}} + \Lambda_{16}\vartheta_{n+\frac{4}{7}} + \Lambda_{17}\vartheta_{n+\frac{5}{7}} + \Lambda_{18}\vartheta_{n+\frac{6}{7}} + \Lambda_{19}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n-\frac{2}{7}} - \rho_n + \frac{2}{7}h\rho'_n - \frac{\left(-\frac{2}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{7}} + \Lambda_{21}\vartheta_{n-\frac{2}{7}} + \Lambda_{22}\vartheta_n + \Lambda_{23}\vartheta_{n+\frac{1}{7}} + \Lambda_{24}\vartheta_{n+\frac{2}{7}} + \Lambda_{25}\vartheta_{n+\frac{3}{7}} + \Lambda_{26}\vartheta_{n+\frac{4}{7}} + \Lambda_{27}\vartheta_{n+\frac{5}{7}} + \Lambda_{28}\vartheta_{n+\frac{6}{7}} + \Lambda_{29}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{1}{7}} - \rho_n - \frac{1}{7}h\rho'_n - \frac{\left(\frac{1}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{7}} + \Lambda_{31}\vartheta_{n-\frac{2}{7}} + \Lambda_{32}\vartheta_n + \Lambda_{33}\vartheta_{n+\frac{1}{7}} + \Lambda_{34}\vartheta_{n+\frac{2}{7}} + \Lambda_{35}\vartheta_{n+\frac{3}{7}} + \Lambda_{36}\vartheta_{n+\frac{4}{7}} + \Lambda_{37}\vartheta_{n+\frac{5}{7}} + \Lambda_{38}\vartheta_{n+\frac{6}{7}} + \Lambda_{39}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{2}{7}} - \rho_n - \frac{2}{7}h\rho'_n - \frac{\left(\frac{2}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{7}} + \Lambda_{41}\vartheta_{n-\frac{2}{7}} + \Lambda_{42}\vartheta_n + \Lambda_{43}\vartheta_{n+\frac{1}{7}} + \Lambda_{44}\vartheta_{n+\frac{2}{7}} + \Lambda_{45}\vartheta_{n+\frac{3}{7}} + \Lambda_{46}\vartheta_{n+\frac{4}{7}} + \Lambda_{47}\vartheta_{n+\frac{5}{7}} + \Lambda_{48}\vartheta_{n+\frac{6}{7}} + \Lambda_{49}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{3}{7}} - \rho_n - \frac{3}{7}h\rho'_n - \frac{\left(\frac{3}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{7}} + \Lambda_{51}\vartheta_{n-\frac{2}{7}} + \Lambda_{52}\vartheta_n + \Lambda_{53}\vartheta_{n+\frac{1}{7}} + \Lambda_{54}\vartheta_{n+\frac{2}{7}} + \Lambda_{55}\vartheta_{n+\frac{3}{7}} + \Lambda_{56}\vartheta_{n+\frac{4}{7}} + \Lambda_{57}\vartheta_{n+\frac{5}{7}} + \Lambda_{58}\vartheta_{n+\frac{6}{7}} + \Lambda_{59}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{4}{7}} - \rho_n - \frac{4}{7}h\rho'_n - \frac{\left(\frac{4}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{7}} + \Lambda_{61}\vartheta_{n-\frac{2}{7}} + \Lambda_{62}\vartheta_n + \Lambda_{63}\vartheta_{n+\frac{1}{7}} + \Lambda_{64}\vartheta_{n+\frac{2}{7}} + \Lambda_{65}\vartheta_{n+\frac{3}{7}} + \Lambda_{66}\vartheta_{n+\frac{4}{7}} + \Lambda_{67}\vartheta_{n+\frac{5}{7}} + \Lambda_{68}\vartheta_{n+\frac{6}{7}} + \Lambda_{69}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{5}{7}} - \rho_n - \frac{5}{7}h\rho'_n - \frac{\left(\frac{5}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{70}\vartheta_{n-\frac{1}{7}} + \Lambda_{71}\vartheta_{n-\frac{2}{7}} + \Lambda_{72}\vartheta_n + \Lambda_{73}\vartheta_{n+\frac{1}{7}} + \Lambda_{74}\vartheta_{n+\frac{2}{7}} + \Lambda_{75}\vartheta_{n+\frac{3}{7}} + \Lambda_{76}\vartheta_{n+\frac{4}{7}} + \Lambda_{77}\vartheta_{n+\frac{5}{7}} + \Lambda_{78}\vartheta_{n+\frac{6}{7}} + \Lambda_{79}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+\frac{6}{7}} - \rho_n - \frac{6}{7}h\rho'_n - \frac{\left(\frac{6}{7}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{80}\vartheta_{n-\frac{1}{7}} + \Lambda_{81}\vartheta_{n-\frac{2}{7}} + \Lambda_{82}\vartheta_n + \Lambda_{83}\vartheta_{n+\frac{1}{7}} + \Lambda_{84}\vartheta_{n+\frac{2}{7}} + \Lambda_{85}\vartheta_{n+\frac{3}{7}} + \Lambda_{86}\vartheta_{n+\frac{4}{7}} + \Lambda_{87}\vartheta_{n+\frac{5}{7}} + \Lambda_{88}\vartheta_{n+\frac{6}{7}} + \Lambda_{89}\vartheta_{n+1}\right)\\
L[\rho(\xi_n);h] &= \rho_{n+1} - \rho_n - h\rho'_n - \frac{(h)^2}{2!}\rho''_n - h^3\left(\Lambda_{90}\vartheta_{n-\frac{1}{7}} + \Lambda_{91}\vartheta_{n-\frac{2}{7}} + \Lambda_{92}\vartheta_n + \Lambda_{93}\vartheta_{n+\frac{1}{7}} + \Lambda_{94}\vartheta_{n+\frac{2}{7}} + \Lambda_{95}\vartheta_{n+\frac{3}{7}} + \Lambda_{96}\vartheta_{n+\frac{4}{7}} + \Lambda_{97}\vartheta_{n+\frac{5}{7}} + \Lambda_{98}\vartheta_{n+\frac{6}{7}} + \Lambda_{99}\vartheta_{n+1}\right)
\end{aligned}\right\}\quad(3.1)$$

$$\left.\begin{aligned}
L[\rho'(\xi_n);h]&=\rho'_{n-\frac{1}{7}}-\rho'_n+\frac{1}{7}h\rho''_n-h^2 f\left(\mathrm{X}_{110}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{111}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{112}\vartheta_n+\mathrm{X}_{113}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{114}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{115}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{116}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{117}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{118}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{119}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n-\frac{2}{7}}-\rho'_n+\frac{2}{7}h\rho''_n-h^2\left(\mathrm{X}_{120}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{121}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{122}\vartheta_n+\mathrm{X}_{123}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{124}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{125}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{126}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{127}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{128}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{129}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+\frac{1}{7}}-\rho'_n-\frac{1}{7}h\rho''_n-h^2\left(\mathrm{X}_{130}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{131}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{132}\vartheta_n+\mathrm{X}_{133}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{134}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{135}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{136}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{137}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{138}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{139}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+\frac{2}{7}}-\rho'_n-\frac{2}{7}h\rho''_n-h^2\left(\mathrm{X}_{140}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{141}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{142}\vartheta_n+\mathrm{X}_{143}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{144}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{145}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{146}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{147}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{148}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{149}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+\frac{3}{7}}-\rho'_n-\frac{3}{7}h\rho''_n-h^2\left(\mathrm{X}_{150}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{151}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{152}\vartheta_n+\mathrm{X}_{153}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{154}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{155}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{156}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{157}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{158}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{159}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+\frac{4}{7}}-\rho'_n-\frac{4}{7}h\rho''_n-h^2\left(\mathrm{X}_{160}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{161}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{162}\vartheta_n+\mathrm{X}_{163}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{164}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{165}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{166}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{167}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{168}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{169}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+\frac{5}{7}}-\rho'_n-\frac{5}{7}h\rho''_n-h^2\left(\mathrm{X}_{170}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{171}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{172}\vartheta_n+\mathrm{X}_{173}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{174}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{175}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{176}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{177}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{178}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{179}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+\frac{6}{7}}-\rho'_n-\frac{5}{7}h\rho''_n-h^2\left(\mathrm{X}_{180}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{181}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{182}\vartheta_n+\mathrm{X}_{183}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{184}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{185}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{186}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{187}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{188}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{189}\vartheta_{n+1}\right)\\
L[\rho'(\xi_n);h]&=\rho'_{n+1}-\rho'_n-h\rho''_n-h^2\left(\mathrm{X}_{190}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{191}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{192}\vartheta_n+\mathrm{X}_{193}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{194}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{195}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{196}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{197}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{198}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{199}\vartheta_{n+1}\right)
\end{aligned}\right\}\tag{3.2}$$

$$\left.\begin{aligned}
L[\rho''(\xi_n);h]&=\rho''_{n-\frac{1}{7}}-\rho''_n-h\left(\mathrm{X}_{210}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{211}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{212}\vartheta_n+\mathrm{X}_{213}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{214}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{215}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{216}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{217}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{218}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{219}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n-\frac{2}{7}}-\rho''_n-h\left(\mathrm{X}_{220}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{221}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{222}\vartheta_n+\mathrm{X}_{223}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{224}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{225}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{226}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{227}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{228}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{229}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+\frac{1}{7}}-\rho''_n-h\left(\mathrm{X}_{230}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{231}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{232}\vartheta_n+\mathrm{X}_{233}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{234}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{235}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{236}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{237}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{238}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{239}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+\frac{2}{7}}-\rho''_n-h\left(\mathrm{X}_{240}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{241}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{242}\vartheta_n+\mathrm{X}_{243}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{244}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{245}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{246}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{247}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{248}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{249}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+\frac{3}{7}}-\rho''_n-h\left(\mathrm{X}_{250}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{251}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{252}\vartheta_n+\mathrm{X}_{253}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{254}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{255}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{256}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{257}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{258}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{259}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+\frac{4}{7}}-\rho''_n-h\left(\mathrm{X}_{260}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{261}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{262}\vartheta_n+\mathrm{X}_{263}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{264}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{265}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{266}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{267}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{268}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{269}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+\frac{5}{7}}-\rho''_n-h\left(\mathrm{X}_{270}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{271}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{272}\vartheta_n+\mathrm{X}_{273}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{274}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{275}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{276}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{277}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{278}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{279}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+\frac{6}{7}}-\rho''_n-h\left(\mathrm{X}_{280}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{281}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{282}\vartheta_n+\mathrm{X}_{283}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{284}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{285}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{286}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{287}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{288}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{289}\vartheta_{n+1}\right)\\
L[\rho''(\xi_n);h]&=\rho''_{n+1}-\rho''_n-h\left(\mathrm{X}_{290}\vartheta_{n-\frac{1}{7}}+\mathrm{X}_{291}\vartheta_{n-\frac{2}{7}}+\mathrm{X}_{292}\vartheta_n+\mathrm{X}_{293}\vartheta_{n+\frac{1}{7}}+\mathrm{X}_{294}\vartheta_{n+\frac{2}{7}}+\mathrm{X}_{295}\vartheta_{n+\frac{3}{7}}+\mathrm{X}_{296}\vartheta_{n+\frac{4}{7}}+\mathrm{X}_{297}\vartheta_{n+\frac{5}{7}}+\mathrm{X}_{298}\vartheta_{n+\frac{6}{7}}+\mathrm{X}_{299}\vartheta_{n+1}\right)
\end{aligned}\right\}\tag{3.3}$$

Corollary 3.2

The local truncation error of (2.6) to (2.8) assume $\rho(\xi)$ to be sufficiently differentiable and expanding equation (3.1) to (3.3) about ξ_n using a Taylor series to obtain

$$\begin{aligned}
&L_{-\frac{1}{7}}[\rho(\xi_n);h]=\left(-1.3423\times10^{-15}\right),\ L_{-\frac{2}{7}}[\rho(\xi_n);h]=\left(-4.9363\times10^{-15}\right),\ L_{\frac{1}{7}}[\rho(\xi_n);h]=\left(-6.2566\times10^{-16}\right),\\
&L_{\frac{2}{7}}[\rho(\xi_n);h]=\left(-4.1514\times10^{-15}\right),\ L_{\frac{3}{7}}[\rho(\xi_n);h]=\left(-9.9180\times10^{-15}\right),\ L_{\frac{4}{7}}[\rho(\xi_n);h]=\left(-1.8499\times10^{-14}\right),\\
&L_{\frac{5}{7}}[\rho(\xi_n);h]=\left(-2.9236\times10^{-14}\right),\ L_{\frac{6}{7}}[\rho(\xi_n);h]=\left(-4.3060\times10^{-14}\right),\ L_1[\rho(\xi_n);h]=\left(-5.9688\times10^{-14}\right),
\end{aligned}$$

$$L_{-\frac{1}{7}}[\rho'(\xi_n);h]=\left(3.4723\times10^{-14}\right),\ L_{-\frac{2}{7}}[\rho'(\xi_n);h]=\left(-7.9947\times10^{-14}\right),\ L_{\frac{1}{7}}[\rho'(\xi_n);h]=\left(-1.3948\times10^{-09}\right),$$
$$L_{\frac{2}{7}}[\rho'(\xi_n);h]=\left(-3.2647\times10^{-14}\right),\ L_{\frac{3}{7}}[\rho'(\xi_n);h]=\left(-5.0341\times10^{-14}\right),\ L_{\frac{4}{7}}[\rho'(\xi_n);h]=\left(-6.7031\times10^{-14}\right),$$
$$L_{\frac{5}{7}}[\rho'(\xi_n);h]=\left(-8.8471\times10^{-14}\right),\ L_{\frac{6}{7}}[\rho'(\xi_n);h]=\left(-8.9137\times10^{-14}\right),\ L_{1}[\rho'(\xi_n);h]=\left(-2.3919\times10^{-13}\right),$$

$$L_{-\frac{1}{7}}[\rho''(\xi_n);h]=\left(-5.5970\times10^{-13}\right),\ L_{-\frac{2}{7}}[\rho''(\xi_n);h]=\left(2.8721\times10^{-12}\right),\ L_{\frac{1}{7}}[\rho''(\xi_n);h]=\left(-1.8069\times10^{-13}\right),$$
$$L_{\frac{2}{7}}[\rho''(\xi_n);h]=\left(-8.4802\times10^{-14}\right),\ L_{\frac{3}{7}}[\rho''(\xi_n);h]=\left(-1.6292\times10^{-13}\right),\ L_{\frac{4}{7}}[\rho''(\xi_n);h]=\left(-6.7031\times10^{-14}\right),$$
$$L_{\frac{5}{7}}[\rho''(\xi_n);h]=\left(-2.4772\times10^{-13}\right),\ L_{\frac{6}{7}}[\rho''(\xi_n);h]=\left(3.1198\times10^{-13}\right),\ L_{1}[\rho''(\xi_n);h]=\left(-3.1199\times10^{-12}\right)$$

Proof

Simplify equations (2.6) to (2.8) with Corollary 3.2 and collect the like terms to obtain

$$L_{-\frac{1}{7}}[\rho(\xi_n);h]=\left(-1.3423\times10^{-15}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),\ L_{-\frac{2}{7}}[\rho(\xi_n);h]=\left(-4.9363\times10^{-15}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),$$
$$L_{\frac{1}{7}}[\rho(\xi_n);h]=\left(-6.2566\times10^{-16}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),\ L_{\frac{2}{7}}[\rho(\xi_n);h]=\left(-4.1514\times10^{-15}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),$$
$$L_{\frac{3}{7}}[\rho(\xi_n);h]=\left(-9.9180\times10^{-15}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),\ L_{\frac{4}{7}}[\rho(\xi_n);h]=\left(-1.8499\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),$$
$$L_{\frac{5}{7}}[\rho(\xi_n);h]=\left(-2.9236\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),\ L_{\frac{6}{7}}[\rho(\xi_n);h]=\left(-4.3060\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),$$
$$L_{1}[\rho(\xi_n);h]=\left(-5.9688\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{13}\right),$$

$$L_{-\frac{1}{7}}[\rho'(\xi_n);h]=\left(3.4723\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),\ L_{-\frac{2}{7}}[\rho'(\xi_n);h]=\left(-7.9947\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),$$
$$L_{\frac{1}{7}}[\rho'(\xi_n);h]=\left(-1.3948\times10^{-09}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),\ L_{\frac{2}{7}}[\rho'(\xi_n);h]=\left(-3.2647\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),$$
$$L_{\frac{3}{7}}[\rho'(\xi_n);h]=\left(-5.0341\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),\ L_{\frac{4}{7}}[\rho'(\xi_n);h]=\left(-6.7031\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),$$
$$L_{\frac{5}{7}}[\rho'(\xi_n);h]=\left(-8.8471\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),\ L_{\frac{6}{7}}[\rho'(\xi_n);h]=\left(-8.9137\times10^{-14}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),$$
$$L_{1}[\rho'(\xi_n);h]=\left(-2.3919\times10^{-13}\right)C_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{12}\right),$$

$$L_{-\frac{1}{7}}[\rho''(\xi_n);h]=\left(-5.5970\times10^{-13}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),\ L_{-\frac{2}{7}}[\rho''(\xi_n);h]=\left(2.8721\times10^{-12}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),$$
$$L_{\frac{1}{7}}[\rho''(\xi_n);h]=\left(-1.8069\times10^{-13}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),\ L_{\frac{2}{7}}[\rho''(\xi_n);h]=\left(-8.4802\times10^{-14}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),$$
$$L_{\frac{3}{7}}[\rho''(\xi_n);h]=\left(-1.6292\times10^{-13}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),\ L_{\frac{4}{7}}[\rho''(\xi_n);h]=\left(-6.7031\times10^{-14}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),$$
$$L_{\frac{5}{7}}[\rho''(\xi_n);h]=\left(-2.4772\times10^{-13}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),\ L_{\frac{6}{7}}[\rho''(\xi_n);h]=\left(3.1198\times10^{-13}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right),$$
$$L_{1}[\rho''(\xi_n);h]=\left(-3.1199\times10^{-12}\right)_{10}h^{10}\rho^{10}(\xi_n)+0\left(h^{11}\right)$$

3.2. Consistency of BHM

Definition 3.1: First and Second Characteristic Polynomials

Given the new numerical scheme, the first and second characteristic polynomials are defined as,

$$\rho(z)=\sum_{j=0}^{1}\alpha_j z^j \qquad (3.4)$$

and

$$\sigma(z)=\sum_{j=0}^{1}\beta_j z^j \qquad (3.5)$$

where z is the principal root, $\alpha_1 \neq 0$ and $\alpha_0^2+\beta_0^2 \neq 0$ (Sabo and Adeyeye, 2025).

The new numerical scheme is said to be consistent if it satisfies the following conditions;

1) the order $p \geq 1$,
2) $\sum_{j=0}^{1}\alpha_j = 0$ and
3) $p'(1)=\sigma(1)$

The LBA is consistent since it is of uniform order ten (According to definition 3.1).

3.3. Zero Stability of BHM

Definition 3.2: A new BHM is said to be zero-stable if the roots $z_s,\ s=1,2,\ldots,n$ of the first characteristic polynomial $\overline{p}(z)$, defined by

$$\overline{p}(z)=\det\left[zA^{(0)}-E\right] \qquad (3.6)$$

satisfies $\left|z_s\right| \leq 1$ and every root with $\left|z_s\right| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $p(z)=z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds (Sabo and Adeyeye, 2025).

By definition 3.2, a BHM is said to be Zero-stable for any well-behaved initial value problem, that is

$$\rho(u)=\frac{32091-1393u+18230u^2+51194u^3-21346u^4+87627u^5-89621u^6-7112u^7+1067u^8}{21109-34812u+98217u^2-91276u^3-124576u^4+9021u^5+1061u^6-990u^7+701u^8+89u^9} \quad (3.7)$$

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{bmatrix}\right|=\begin{bmatrix}z&0&0&0&0&0&0&-1\\0&z&0&0&0&0&0&-1\\0&0&z&0&0&0&0&-1\\0&0&0&z&0&0&0&-1\\0&0&0&0&z&0&0&-1\\0&0&0&0&0&z&0&-1\\0&0&0&0&0&0&z&-1\\0&0&0&0&0&0&0&z-1\end{bmatrix}=z^7(z-1)$$

Thus, solving for z in

$$z^7(z-1)=0 \qquad (3.8)$$

Solving for (3.8), gives $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = 0$ and $z_8 = 1$. Hence, the BHM is zero-stable.

3.4. Convergence of BHM

The LBA is convergent, since it consistent and zero-stable by theorem that stated “the necessary and sufficient conditions for new BHM to be convergent is that it must be consistent and zero-stable” (Tumba et al. 2021).

3.5. Region of Absolute Stability (RAS) of BHM

To determine the regions of absolute stability of new BHM, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted (Tumba et al. 2021). This method is called the Boundary Locus Method (BLM). The BLM was Apply on BHM in order to obtain the stability polynomial of the form

$$\left.\begin{array}{l}\bar{h}(\pi) = \left(1.2868137873955471403\mathrm{0} \times 10^{-10}\pi^8 + 1.6427495712048782481\mathrm{0} \times 10^{-11} w^9\right)h^9 \\ + \left(3.1248421845693796147\mathrm{0} \times 10^{-09}\pi^8 + 1.0958430766456317025\mathrm{0} \times 10^{-09} w^9\right)h^8 \\ \left(-2.9386189178137208371\mathrm{0} \times 10^{08}\pi^9 - 6.1210944260778762775\mathrm{0} \times 10^{-08}\pi^8\right)h^7 \\ + \left(-3.5708535459166472973\mathrm{0} \times 10^{-07}\pi^9 - 0.0000022360104258002\mathrm{3}\pi^8\right)h^6 \\ + \left(0.0000161568374025800\mathrm{6}\pi^9 + 0.0000034316602701940\mathrm{1}\pi^8\right)h^5 + \\ \left(0.00067180049138712778\pi^8 - 0.00006213327629717421\pi^9\right)h^4 \\ + \left(-0.00330853223669205291\pi^9 + 0.00568518063366523318\pi^8\right)h^3 + \\ \left(0.05497730749163800063\pi^9 - 0.01918674689758213849\pi^8\right)h^2 \\ + \left(-0.35714285714285714286\pi^9 - 0.35714285714285714286\pi^8\right)h - \pi^8 + \pi^9\end{array}\right\} \quad (3.9)$$

The RAS of LBA was obtained by using the stability polynomial (3.9) on Matlab R2024a as

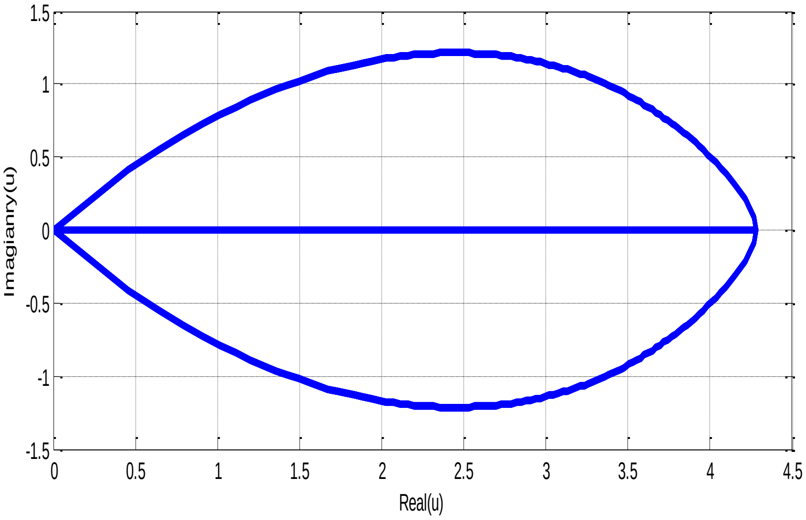

Figure 3.1. Regions of Absolute Stability of BHM

4. Numerical Examples

The numerical examples of the new BHM were carryout by solving the new BHM on linear and nonlinear Volterra Integro-Differential Equations (VIDEs) of second kind was carryout. The new BHM was applied on four (VIDEs) of second kind problems and the results are numerically tabulated and textually shown to compared the results with the existing methods in terms of error.

Problem 4.1

The linear Volterra integro-differential equation of the form

$$\rho'(\xi)=1-\int_0^{\xi}\rho(\tau)d\tau,\ \rho(0)=0,\ 0\le\xi\le1 \qquad (4.1)$$

is considered with exact solution given by

$$\rho(\xi)=\sin(\xi) \qquad (4.2)$$

Source: [Faires and Burden, (2015), Majid and Mohamed, (2019)].

Problem 4.2

The linear Volterra integro-differential equation of the form

$$\rho'(\xi)=1+\int_0^{\xi}\rho(\tau)d\tau,\ \rho(0)=0,\ 0\le\xi\le1 \qquad (4.3)$$

is consider with exact solution given by

$$\rho(\xi)=\sin h(\xi) \qquad (4.4)$$

Source [Rabiel et al., (2019), Olowe *et al.*, (2023)]

Problem 4.3

The nonlinear Volterra integro-differential equation of the form

$$\rho'(\xi)=\xi\exp(1-\rho(\xi))-\frac{1}{(1+\xi)^2}-\xi-\int_0^{\xi}\frac{1}{(1+\tau)^2}\exp(1-\rho(\xi))d\tau,\ \rho(0)=1 \qquad (4.5)$$

is consider with exact solution given by

$$\rho(\xi)=\frac{1}{1+\xi} \qquad (4.6)$$

Source: [Faires and Burden, (2015), Majid and Mohamed, (2019)]

Problem 4.4

The nonlinear Volterra integro-differential equation of the form

$$\rho''(\xi)=\left(\frac{1}{2}\exp(-\xi)\sin(\xi)-\sin(\xi)\right)+\int_0^{\xi}\exp(\tau)\sin(\xi)\rho'(\tau)d\tau+\rho(\xi),\ \rho(0)=-1,\ \rho'(0)=1 \qquad (4.7)$$

is consider with exact solution given by

$$\rho(\xi)=\sin(\xi)-\cos(\xi) \qquad (4.8)$$

Source: [Kamoh *et al.*, (2017)]

The following Acronyms were used in the tables, figures, discussion of results and others.

Acronyms	**Meaning**
ξ	Points of Evaluation
AS	Analytic Solution
CNR	Computed Numerical Results in new BHM
EBHM	Error in new BHM
EABM5	Error in Fifth Order Adams-Bashforth-Moulton Predictor-Corrector Method developed by Faires and Burden (2015)
E2P3B	Error in Two Point Three-Step Block Method developed by Majid and Mohamed, (2019)
ETFSM	Trigonometrically Fitted Simpson's Method developed by Olowe *et al.,* (2023)
EGLM	Third Order General Linear Method developed by Rabiel *et al.*, (2019)
ETR	Trapezoidal Rule using Three Quadrature Rules developed by Kamoh *et al.,* (2017)
EGR	Gussian Rule using Three Quadrature Rules developed by Kamoh *et al.,* (2017)
ES13R	Simpsons 1/3 Rule using Three Quadrature Rules developed by Kamoh *et al.,* (2017)

Table 4.1. Comparative Analysis of Numerical Results for Example 4.1

ξ	**AS**	**SNR**	**EBHM**	**EABM5**	**E2P3B**
0.025	0.02499739591471233066	0.02499739591471233066	0.0000e00	4.4529e-09	1.2349e-09
0.0125	0.01249967448170978872	0.01249967448170978872	0.0000e00	2.3862e-10	3.8642e-11
0.00625	0.00624995930997530612	0.00624995930997530612	0.0000e00	1.4271e-11	1.2080e-12
0.003125	0.00312499491373946269	0.00312499491373946269	0.0000e00	8.6009e-13	3.7751e-14
0.0015625	0.00156249936421720001	0.00156249936421720001	0.0000e00	4.7296e-14	5.3291e-15
0.00078125	0.00078124992052714272	0.00078124992052714272	0.0000e00	1.6764e-14	1.3545e-14

Table 4.2. Comparative Analysis of Numerical Results for Example 4.2

ξ	**AS**	**SNR**	**EBHM**	**EGLM**	**ETFSM**
0.1	0.10016675001984402582	0.10016675001984402150	0.0000e00	1.4606e-06	3.7864e-12
0.025	0.02500260424804808603	0.02500260424804808603	0.0000e00	1.6319e-08	2.2030e-16
0.01	0.01000016666750000198	0.01000016666750000198	0.0000e00	8.3870e-10	3.5789e-19
0.005	0.00500002083335937502	0.00500002083335937502	0.0000e00	1.7077e-11	8.0866e-18
0.001	0.00100000016666667500	0.00100000016666667500	0.0000e00	7.2935e-13	-

Table 4.3. Comparative Analysis of Numerical Results for Example 4.3

ξ	**AS**	**SNR**	**EBHM**	**EABM5**	**E2P3B**
0.025	0.97560975609756097561	0.97560975609756097732	1.7100e-18	1.7212e-08	8.3237e-08
0.0125	0.98765432098765432099	0.98765432098765432186	8.7000e-19	2.0551e-09	3.8384e-09
0.00625	0.99378881987577639752	0.99378881987577639719	3.2000e-19	1.9089e-10	2.0775e-10
0.003125	0.99688473520249221184	0.99688473520249221180	4.0000e-20	1.1926e-11	1.2654e-11
0.0015625	0.99843993759750390016	0.99843993759750390017	1.0000e-20	7.4529e-13	9.6889e-13
0.00078125	0.99921935987509758002	0.99921935987509758002	0.0000e00	4.6518e-14	4.3676e-13

Table 4.4. Comparative Analysis of Numerical Results for Example 4.4

ξ	**AS**	**SNR**	**EBHM**	**ETR**	**EGR**	**ES13R**
0.16	-0.82790907676138098573	-0.82790907391209126578	2.8493e-08	1.9000e-09	1.2512e-05	3.4135e-06
0.32	-0.63466885746632310092	-0.63466885129083655378	6.1755e-09	1.3900e-08	3.2919e-04	2.6456e-05
0.48	-0.42521574723780130526	-0.42521574797120933784	7.3341e-10	5.6600e-08	2.0860e-03	8.5434e-05
0.64	-0.20490031652190056171	-0.20490031629875309746	2.2315e-10	1.5330e-07	7.7089e-03	1.9242e-04
0.80	0.02064938155235734071	0.02064938159823650947	4.5879e-11	3.2662e-07	2.1169e-02	3.5240e-04
0.96	0.24567158222854160950	0.24567158224987091266	2.1329e-11	5.9210e-07	4.8093e-02	5.6512e-04

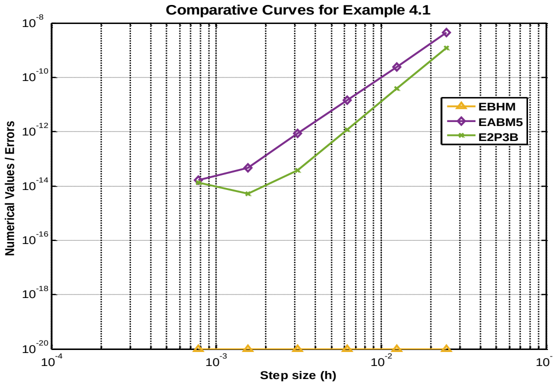

Figure 4.1. Graphical Representation of Numerical Results in Table 4.1

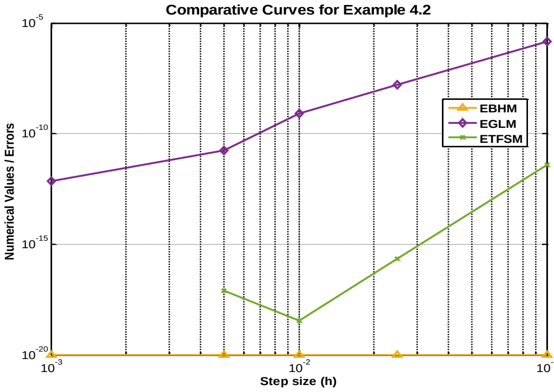

Figure 4.2. Graphical Representation of Numerical Results in Table 4.2

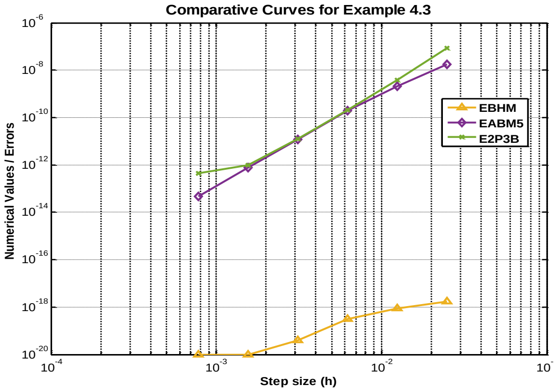

Figure 4.3. Graphical Representation of Numerical Results in Table 4.3

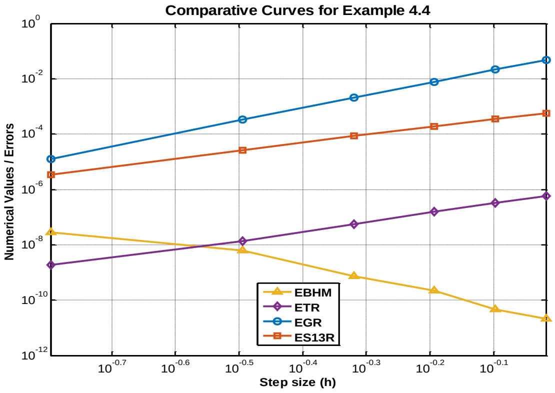

Figure 4.4. Graphical Representation of Numerical Results in Table 4.4

5. Discussion of Results

This study developed a Block Hybrid Method (BHM) through third-derivative Linear Block Algorithm (LBA) for the numerical integration of second-kind Volterra integro-differential equations. The derivation followed the framework of general linear multistep methods, as outlined in Adeyeye and Omar (2016, 2019), with the assistance of Scientific Workplace 5.5 for symbolic computations.

The BHM was constructed by expanding the general linear block formulation to include higher-order derivatives, yielding continuous polynomial schemes for the numerical solution. Unknown coefficients in the scheme and its derivatives were determined systematically, as detailed in the appendix, ensuring that the algorithm could accurately approximate both the differential and integral components of the VIDEs. This approach allowed the new BHM to handle complex integral-differential structures while maintaining computational efficiency.

The algorithm's fundamental properties were rigorously analyzed to ensure reliability and accuracy. The order and error constant were established through Taylor series expansions, demonstrating the method's high precision. Consistency was confirmed via characteristic polynomials, showing that the BHM maintains uniform order ten. Zero-stability was proven by evaluating the roots of the characteristic polynomial, ensuring controlled propagation of errors during integration.

Consequently, convergence follows naturally from the method's consistency and zero-stability. Finally, the region of absolute stability was determined using the Boundary Locus Method and visualized in MATLAB, confirming the BHM's robustness for practical applications. Overall, these analyses verify that the new LBA is both theoretically sound and practically effective for solving second-kind VIDEs.

The numerical experiments carried out using the new BHM demonstrate its effectiveness in solving linear and nonlinear Volterra Integro-Differential Equations (VIDEs) of the second kind. Four benchmark problems (Problems 4.1-4.4) were considered, and the results were both tabulated and graphically represented for comparison. The computed numerical results from the new BHM were evaluated against analytic solutions and existing numerical methods, including Adams-Bashforth-Moulton Predictor-Corrector, Two-Point Three-Step Block Method, Trigonometrically Fitted Simpson's Method, and other quadrature-based schemes. The comparison focused primarily on the error values, providing a quantitative assessment of accuracy and stability.

For Problem 4.1, the results in Table 4.1 and Figure 4.1 show that the new BHM produces numerical solutions that closely match the analytic solutions across various step sizes. The associated error (EBHM) is negligible, often smaller than those from the existing methods like EABM5 and E2P3B, indicating superior precision of the new algorithm. As the step size decreases, the error consistently diminishes, reflecting the method's convergence behavior and high order of accuracy. This trend confirms the robustness of the BHM in capturing the dynamics of linear VIDEs with minimal computational discrepancy.

Problem 4.2, detailed in Table 4.2 and Figure 4.2, highlights the performance of the BHM on a different class of VIDEs with distinct characteristics. Again, the computed results align very closely with the exact solution, with ELBA values approaching zero. Compared to methods such as EGLM and ETFSM, the new BHM demonstrates higher accuracy, particularly for smaller step sizes where other methods show minor deviations. This emphasizes the efficiency of the new BHM in handling both the integral and differential components of VIDEs with consistent precision.

Problems 4.3 and 4.4, summarized in Tables 4.3 and 4.4 and illustrated in Figures 4.3 and 4.4, further confirm the capability of the new BHM in solving both linear and nonlinear VIDEs. Across all evaluated step sizes, the BHM maintains very low errors relative to established methods, including EABM5, E2P3B, ETR, and ES13R. Particularly for Problem 4.4, a nonlinear case, the new BHM manages to control numerical errors effectively where other quadrature-based methods exhibit higher deviations. These results indicate that the new BHM is not only accurate but also reliable and versatile, making it a strong candidate for solving a wide range of Volterra integro-differential equations.

░6. Summary and Conclusion

This study developed BHM through third-derivative Linear Block Algorithm (LBA) for the numerical treatment of Volterra integro-differential equations of the second kind. The method was formulated within the framework of general linear multistep schemes and constructed as a single-step block hybrid algorithm capable of computing solutions at multiple points simultaneously. By incorporating higher-order derivatives, continuous polynomial approximations were obtained, allowing accurate and efficient handling of both the differential and integral components of the equations. A comprehensive theoretical analysis confirmed that the new BHM is of uniform order ten, consistent, zero-stable, and convergent. The region of absolute stability, determined using the Boundary Locus Method, further demonstrated the robustness of the algorithm for practical numerical integration.

The numerical experiments carried out on several linear and nonlinear benchmark problems showed that the new BHM produces results that agree closely with exact solutions and significantly outperform several existing methods in terms of accuracy and error control. Across varying step sizes, the method consistently yielded very small errors, highlighting its efficiency and reliability. In conclusion, the new BHM provides an accurate and stable numerical tool for solving second-kind Volterra integro-differential equations and represents a meaningful contribution to numerical methods for integro-differential problems, with potential for extension to more complex and advanced applications.

░7. Suggestion for Future Research

Based on the findings of this research, the following suggestions for future research are made:

1) The newly developed BHM should be applied to real-world problems such as heat transfer with memory, viscoelasticity, epidemiological models and population dynamics to further explore their effectiveness.

2) Future studies may extend the developed BHM to nonlinear integral and integro-differential equations, which more accurately represent real-life scientific and engineering problems.

3) Further research could focus on the numerical solution of systems of Volterra and Fredholm integral equations arising from coupled physical and engineering models.

4) The application of the new numerical schemes to fractional-order integral equations may be explored to account for memory and hereditary effects in complex systems.

5) Future work may investigate the effectiveness of the BHM in solving singular and weakly singular integral equations.

6) Extension of the new BHM to Volterra–Fredholm integro-differential equations is recommended to broaden their applicability.

Declarations

Source of Funding

This study did not receive any grant from funding agencies in the public, commercial, or not–for–profit sectors.

Competing Interests Statement

The authors have not declared any conflict of interest.

Consent for publication

The authors declare that they consented to the publication of this study.

Authors' contributions

Both the authors took part in literature review, analysis, and manuscript writing equally.

Informed Consent

Not applicable for this study.

Availability of data and material

Supplementary information is available from the authors upon reasonable request.

Institutional Review Board Statement

Not applicable for this study.

Ethical Approval

Not applicable for this study.

References

Adeyeye, O., & Omar, Z. (2019). Direct solution of initial and boundary value problems of third order ODEs using maximal-order fourth-derivative block method. AIP Conference Proceedings, 2164: 020014. https://doi.org/10.1063/1.5121039.

Ahmed, F.S. (2020). Approximate solutions for systems of Volterra integro-differential equations using Laplace–Adomian method. Iraqi Journal of Science, 61(10): 2655–2662. https://doi.org/10.24996/ijs.2020.61. 10.22.

Ajileye, G., & Amoo, S.A. (2023). Numerical solution to Volterra integro-differential equations using collocation approximation. Mathematics and Computational Sciences, 4(1): 1–8. https://doi.org/10.30511/mcs.2023.1978083.1099.

Akbar, M., Nawaz, R., & Ahsan, S. (2019). Optimum solutions of Volterra and Fredholm integro-differential equations. International Journal of Theoretical and Applied Mathematics, 5(6): 100–112. https://doi.org/10.11648/j.ijtam.20190506.14.

Aloko, M.D. (2018). New approximate methods for solving nonlinear Volterra–Fredholm integro-differential equations of the second kind. PhD Dissertation, University of Lagos, Nigeria.

Anakira, N.R., et al. (2023). An algorithm for solving linear and non-linear Volterra integro-differential equations. International Journal of Advanced Soft Computing Applications, 15(3). https://doi.org/10.15849/ijasca.231130.05.

Faires, D., & Burden, R.L. (2015). Numerical analysis. Belmont, CA: Thomson Brooks/Cole.

Falade, K.I., & Tiamiyu, A. (2020). Computational algorithm for the numerical solution of systems of Volterra integro-differential equations. Academic Journal of Applied Mathematical Sciences, 6(6): 66–76.

Ishak, F., & Selamat, M.N.F. (2020). New development of extended trapezoidal method for solving first order linear Volterra integro-differential equations. ASM Science Journal, 13. https://doi.org/10.32802/asmscj.2020.sm26(5.24).

Issa, M.B., & Hamoud, A.A. (2020). Solving systems of the Volterra integro-differential equations using semi-analytical techniques. International Review of Knowledge University, 62(3): 579–582.

Kamoh, N.M., Aboiyar, T., & Onah, E.S. (2017). On investigating some quadrature rules for the solution of second order Volterra integro-differential equations. IOSR Journal of Mathematics, 13(5): 45–50.

Majid, Z.A., & Mohamed, N.A. (2019). Fifth order multistep block method for solving Volterra integro-differential equations of second kind. Sains Malaysiana, 48(3): 677–684. https://doi.org/10.17576/jsm-20 19-4803-22.

Matinfar, M., & Riahifar, A. (2015). Analytical-approximate solution for nonlinear Volterra integro-differential equations. Journal of Linear Topological Algebra, 4(3): 217–228.

Mohammed, J.K., & Khudair, A.R. (2023). Solving Volterra integral equations via fourth-degree hat functions. Partial Differential Equations in Applied Mathematics, 7: 100494. https://doi.org/10.1016/j.padiff.2023. 100494.

Okai, J.O., Ilejimi, D.O., & Musa, I. (2019). Solution of linear Volterra integro-differential equation of the second kind using the modified Adomian decomposition method. Global Scientific Journal, 7(5): 288–294.

Olowe, R.A., et al. (2023). Trigonometrically-fitted Simpson’s method for solving Volterra integro-differential equations. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 8(2): 68–78. https://doi.org/10.6084/m9.figshare.22015772.

Olowe, R.A., et al. (2024). A trigonometrically fitted method for solving Volterra integro-differential equations. Journal of the Nigerian Mathematical Society, 43(2): 203–220.

Rabiel, F., et al. (2019). Numerical solutions of Volterra integro-differential equations using general linear method. Numerical Algebra, Control and Optimization, 9(4): 433–444. https://doi.org/10.3934/naco.2019042.

Rani, D., & Mishra, D. (2019). Solution of Volterra integral and integro-differential equations using modified Laplace–Adomian decomposition methods. Journal of Applied Mathematics and Statistical Inference, 15(1).

Raymond, D., et al. (2023). Numerical application of higher order linear block method for solving some fourth order initial value problems. African Scientific Reports, 2(1): 67. https://doi.org/10.46481/asr.2022.2.1.67.

Sabo, J., & Adeyeye, O. (2025). Analysis of a block method developed with six generalised grid points for solving fourth order initial value problems. Journal of Mathematical Analysis and Modeling, 6(1): 155–173. https://doi.org/10.48185/jmam.v6i1.1361.

Tumba, P., Skwame, Y., & Raymond, D. (2021). Half-step implicit linear hybrid block approach of order four for solving third order ordinary differential equations. Dutse Journal of Pure and Applied Sciences, 7(2b): 124–133.

Appendix

$$\begin{pmatrix}\Lambda_{10}\\ \Lambda_{11}\\ \Lambda_{12}\\ \Lambda_{13}\\ \Lambda_{14}\\ \Lambda_{15}\\ \Lambda_{16}\\ \Lambda_{17}\\ \Lambda_{18}\\ \Lambda_{19}\end{pmatrix}=\begin{pmatrix}-\frac{460423}{7302113280}\\ \frac{781531}{328595097600}\\ -\frac{14986661}{273829248000}\\ \frac{194431}{829785600}\\ -\frac{736139}{3651056640}\\ \frac{7755449}{547658496000}\\ -\frac{398609}{5476584960}\\ \frac{231697}{9127641600}\\ -\frac{586007}{1095316992000}\\ \frac{33769}{65719019520}\end{pmatrix},\begin{pmatrix}\Lambda_{20}\\ \Lambda_{21}\\ \Lambda_{22}\\ \Lambda_{23}\\ \Lambda_{24}\\ \Lambda_{25}\\ \Lambda_{26}\\ \Lambda_{27}\\ \Lambda_{28}\\ \Lambda_{29}\end{pmatrix}=\begin{pmatrix}-\frac{11989}{7779240}\\ \frac{4141}{256714920}\\ -\frac{321631}{106964550}\\ \frac{132563}{106964550}\\ -\frac{209999}{213929100}\\ \frac{27971}{42785820}\\ -\frac{6931}{21392910}\\ \frac{11771}{106964550}\\ -\frac{9727}{427858200}\\ \frac{2759}{1283574600}\end{pmatrix},\begin{pmatrix}\Lambda_{30}\\ \Lambda_{31}\\ \Lambda_{32}\\ \Lambda_{33}\\ \Lambda_{34}\\ \Lambda_{35}\\ \Lambda_{36}\\ \Lambda_{37}\\ \Lambda_{38}\\ \Lambda_{39}\end{pmatrix}=\begin{pmatrix}-\frac{1348153}{1095316992000}\\ \frac{50471}{65719019520}\\ -\frac{9326819}{273829248000}\\ \frac{184957}{782369280}\\ -\frac{7311389}{547658496000}\\ \frac{911717}{10953169920}\\ -\frac{1105133}{273829248000}\\ \frac{33841}{2489356800}\\ -\frac{61331}{21906339840}\\ \frac{86909}{328595097600}\end{pmatrix},\begin{pmatrix}\Lambda_{40}\\ \Lambda_{41}\\ \Lambda_{42}\\ \Lambda_{43}\\ \Lambda_{44}\\ \Lambda_{45}\\ \Lambda_{46}\\ \Lambda_{47}\\ \Lambda_{48}\\ \Lambda_{49}\end{pmatrix}=\begin{pmatrix}-\frac{11321}{142619400}\\ \frac{6451}{1283574600}\\ -\frac{195709}{106964550}\\ \frac{6199}{2376990}\\ -\frac{1691}{2037420}\\ \frac{117239}{213929100}\\ -\frac{28603}{106964550}\\ \frac{3217}{35654850}\\ -\frac{1591}{85571640}\\ \frac{41}{23337720}\end{pmatrix},\begin{pmatrix}\Lambda_{50}\\ \Lambda_{51}\\ \Lambda_{52}\\ \Lambda_{53}\\ \Lambda_{54}\\ \Lambda_{55}\\ \Lambda_{56}\\ \Lambda_{57}\\ \Lambda_{58}\\ \Lambda_{59}\end{pmatrix}=\begin{pmatrix}-\frac{7587}{38635520}\\ \frac{303}{24586240}\\ -\frac{215757}{48294400}\\ \frac{2881503}{338060800}\\ -\frac{351189}{676121600}\\ \frac{24723}{19317760}\\ -\frac{41841}{67612160}\\ \frac{71091}{338060800}\\ -\frac{58887}{1352243200}\\ \frac{5583}{1352243200}\end{pmatrix},$$

$$\begin{pmatrix}\Lambda_{60}\\ \Lambda_{61}\\ \Lambda_{62}\\ \Lambda_{63}\\ \Lambda_{64}\\ \Lambda_{65}\\ \Lambda_{66}\\ \Lambda_{67}\\ \Lambda_{68}\\ \Lambda_{69}\end{pmatrix}=\begin{pmatrix}-\frac{3928}{10696455}\\ \frac{3692}{160446825}\\ -\frac{40232}{4862025}\\ \frac{959584}{53482275}\\ -\frac{24392}{10696455}\\ \frac{201136}{53482275}\\ -\frac{248}{218295}\\ \frac{21088}{53482275}\\ -\frac{4364}{53482275}\\ \frac{248}{32089365}\end{pmatrix},\begin{pmatrix}\Lambda_{70}\\ \Lambda_{71}\\ \Lambda_{72}\\ \Lambda_{73}\\ \Lambda_{74}\\ \Lambda_{75}\\ \Lambda_{76}\\ \Lambda_{77}\\ \Lambda_{78}\\ \Lambda_{79}\end{pmatrix}=\begin{pmatrix}-\frac{858875}{1460422656}\\ \frac{482875}{13143803904}\\ -\frac{14492875}{1095316992}\\ \frac{11291125}{365105664}\\ -\frac{494875}{66382848}\\ \frac{20944625}{2190633984}\\ -\frac{449125}{1095316992}\\ \frac{234625}{365105664}\\ -\frac{560375}{4381267968}\\ \frac{159125}{13143803904}\end{pmatrix},\begin{pmatrix}\Lambda_{80}\\ \Lambda_{81}\\ \Lambda_{82}\\ \Lambda_{83}\\ \Lambda_{84}\\ \Lambda_{85}\\ \Lambda_{86}\\ \Lambda_{87}\\ \Lambda_{88}\\ \Lambda_{89}\end{pmatrix}=\begin{pmatrix}-\frac{4563}{5282200}\\ \frac{57}{1056440}\\ -\frac{25569}{1320550}\\ \frac{12519}{264110}\\ -\frac{40041}{2641100}\\ \frac{9747}{528220}\\ -\frac{387}{120050}\\ \frac{3051}{1320550}\\ -\frac{171}{1056440}\\ \frac{93}{5282200}\end{pmatrix},\begin{pmatrix}\Lambda_{90}\\ \Lambda_{91}\\ \Lambda_{92}\\ \Lambda_{93}\\ \Lambda_{94}\\ \Lambda_{95}\\ \Lambda_{96}\\ \Lambda_{97}\\ \Lambda_{98}\\ \Lambda_{99}\end{pmatrix}=\begin{pmatrix}-\frac{54313}{45619200}\\ \frac{10171}{136857600}\\ -\frac{303899}{11404800}\\ \frac{153811}{2280960}\\ -\frac{115199}{4561920}\\ \frac{699769}{22809600}\\ -\frac{107947}{11404800}\\ \frac{81011}{11404800}\\ -\frac{959}{829440}\\ \frac{1321}{27371520}\end{pmatrix}$$

$$\begin{pmatrix} X_{110}\\ X_{111}\\ X_{112}\\ X_{113}\\ X_{114}\\ X_{115}\\ X_{116}\\ X_{117}\\ X_{118}\\ X_{119} \end{pmatrix} = \begin{pmatrix} -\frac{530113}{260789760}\\ \frac{190073}{2933884800}\\ \frac{10946503}{977961600}\\ -\frac{5632757}{977961600}\\ \frac{21995}{4346496}\\ -\frac{7012483}{1955923200}\\ \frac{120719}{65197440}\\ -\frac{211151}{325987200}\\ \frac{4181}{30561300}\\ -\frac{30887}{2347107840} \end{pmatrix}, \begin{pmatrix} X_{120}\\ X_{121}\\ X_{122}\\ X_{123}\\ X_{124}\\ X_{125}\\ X_{126}\\ X_{127}\\ X_{128}\\ X_{129} \end{pmatrix} = \begin{pmatrix} \frac{72151}{3056130}\\ \frac{8501}{7334712}\\ \frac{129758}{7640325}\\ \frac{4124}{7640325}\\ \frac{112403}{30561300}\\ -\frac{5899}{1528065}\\ \frac{3676}{1528065}\\ -\frac{7192}{7640325}\\ \frac{1871}{8731800}\\ -\frac{997}{45841950} \end{pmatrix}, \begin{pmatrix} X_{130}\\ X_{131}\\ X_{132}\\ X_{133}\\ X_{134}\\ X_{135}\\ X_{136}\\ X_{137}\\ X_{138}\\ X_{139} \end{pmatrix} = \begin{pmatrix} -\frac{1030627}{3911846400}\\ \frac{19619}{1173553920}\\ \frac{6050587}{977961600}\\ \frac{171137}{27941760}\\ -\frac{1513637}{488980800}\\ \frac{148619}{78236928}\\ -\frac{894727}{977961600}\\ \frac{300263}{977961600}\\ -\frac{24677}{391184640}\\ \frac{69823}{11735539200} \end{pmatrix}, \begin{pmatrix} X_{140}\\ X_{141}\\ X_{142}\\ X_{143}\\ X_{144}\\ X_{145}\\ X_{146}\\ X_{147}\\ X_{148}\\ X_{149} \end{pmatrix} = \begin{pmatrix} -\frac{1648}{2546775}\\ \frac{7453}{183367800}\\ \frac{15791}{1091475}\\ \frac{43954}{1528065}\\ -\frac{251}{58212}\\ \frac{30158}{7640325}\\ -\frac{1699}{848925}\\ \frac{194}{282975}\\ \frac{1741}{12224520}\\ \frac{62}{4584195} \end{pmatrix}, \begin{pmatrix} X_{150}\\ X_{151}\\ X_{152}\\ X_{153}\\ X_{154}\\ X_{155}\\ X_{156}\\ X_{157}\\ X_{158}\\ X_{159} \end{pmatrix} = \begin{pmatrix} -\frac{9729}{9658880}\\ \frac{19}{301840}\\ \frac{272319}{12073600}\\ \frac{648267}{12073600}\\ \frac{128529}{12073600}\\ \frac{1143}{137984}\\ \frac{7923}{2414720}\\ \frac{13131}{12073600}\\ -\frac{2697}{12073600}\\ \frac{1019}{48294400} \end{pmatrix}, \begin{pmatrix} X_{160}\\ X_{161}\\ X_{162}\\ X_{163}\\ X_{164}\\ X_{165}\\ X_{166}\\ X_{167}\\ X_{168}\\ X_{169} \end{pmatrix} = \begin{pmatrix} -\frac{2092}{1528065}\\ \frac{1954}{22920975}\\ \frac{234296}{7640325}\\ \frac{599216}{7640325}\\ \frac{8524}{305613}\\ \frac{219992}{7640325}\\ -\frac{496}{218295}\\ \frac{9904}{7640325}\\ -\frac{2146}{7640325}\\ \frac{124}{4584195} \end{pmatrix},$$

$$\begin{pmatrix} X_{170}\\ X_{171}\\ X_{172}\\ X_{173}\\ X_{174}\\ X_{175}\\ X_{176}\\ X_{177}\\ X_{178}\\ X_{179} \end{pmatrix} = \begin{pmatrix} -\frac{3375}{1931776}\\ \frac{25625}{234710784}\\ \frac{1519675}{39118464}\\ \frac{4028875}{39118464}\\ \frac{147625}{3259872}\\ \frac{4004375}{78236928}\\ \frac{198875}{13039488}\\ \frac{46925}{13039488}\\ -\frac{5125}{11176704}\\ \frac{18875}{469421568} \end{pmatrix}, \begin{pmatrix} X_{180}\\ X_{181}\\ X_{182}\\ X_{183}\\ X_{184}\\ X_{185}\\ X_{186}\\ X_{187}\\ X_{188}\\ X_{189} \end{pmatrix} = \begin{pmatrix} -\frac{387}{188650}\\ \frac{19}{150920}\\ \frac{4404}{94325}\\ \frac{2424}{18865}\\ \frac{23013}{377300}\\ \frac{285}{3773}\\ \frac{3126}{94325}\\ \frac{2196}{94325}\\ \frac{171}{150920}\\ \frac{1}{188650} \end{pmatrix}, \begin{pmatrix} X_{190}\\ X_{191}\\ X_{192}\\ X_{193}\\ X_{194}\\ X_{195}\\ X_{196}\\ X_{187}\\ X_{198}\\ X_{199} \end{pmatrix} = \begin{pmatrix} -\frac{32431}{11404800}\\ \frac{1631}{8553600}\\ -\frac{161857}{2851200}\\ \frac{84329}{570240}\\ \frac{50029}{570240}\\ \frac{490931}{5702400}\\ \frac{179879}{2851200}\\ \frac{107093}{2851200}\\ \frac{6349}{285120}\\ \frac{1723}{1368576} \end{pmatrix}, \begin{pmatrix} X_{210}\\ X_{211}\\ X_{212}\\ X_{213}\\ X_{214}\\ X_{215}\\ X_{216}\\ X_{217}\\ X_{218}\\ X_{219} \end{pmatrix} = \begin{pmatrix} -\frac{2655563}{50803200}\\ \frac{1169}{1036800}\\ -\frac{859009}{6350400}\\ \frac{274849}{3175200}\\ -\frac{397331}{5080320}\\ -\frac{1424417}{25401600}\\ \frac{92567}{3175200}\\ \frac{65039.}{6350400}\\ \frac{110219}{50803200}\\ \frac{425}{2032128} \end{pmatrix}, \begin{pmatrix} X_{220}\\ X_{221}\\ X_{222}\\ X_{223}\\ X_{224}\\ X_{225}\\ X_{226}\\ X_{227}\\ X_{228}\\ X_{229} \end{pmatrix} = \begin{pmatrix} -\frac{37829}{158760}\\ -\frac{3956}{99225}\\ \frac{34369}{396900}\\ -\frac{45331}{198450}\\ \frac{103987}{396900}\\ -\frac{83291}{396900}\\ \frac{9239}{79380}\\ -\frac{8467}{198450}\\ \frac{3697}{396900}\\ -\frac{9}{9800} \end{pmatrix}, \begin{pmatrix} X_{230}\\ X_{231}\\ X_{232}\\ X_{233}\\ X_{234}\\ X_{235}\\ X_{236}\\ X_{237}\\ X_{238}\\ X_{229} \end{pmatrix} = \begin{pmatrix} \frac{163531}{50803200}\\ \frac{425}{2032128}\\ \frac{391711}{6350400}\\ \frac{349817}{3175200}\\ \frac{1083167}{25401600}\\ \frac{129581}{5080320}\\ -\frac{38599}{3175200}\\ \frac{25759}{6350400}\\ \frac{42187}{50803200}\\ \frac{1}{12800} \end{pmatrix},$$

$$\begin{pmatrix} X_{240}\\ X_{241}\\ X_{242}\\ X_{243}\\ X_{244}\\ X_{245}\\ X_{246}\\ X_{247}\\ X_{248} \end{pmatrix} = \begin{pmatrix} -\frac{1769}{793800}\\ \frac{13}{99225}\\ \frac{21809}{396900}\\ \frac{7193}{39690}\\ \frac{20291}{396900}\\ \frac{1013}{396900}\\ \frac{1213}{396900}\\ \frac{253}{198450}\\ \frac{23}{79380}\\ \frac{23}{793800} \end{pmatrix}, \begin{pmatrix} X_{250}\\ X_{251}\\ X_{252}\\ X_{253}\\ X_{254}\\ X_{255}\\ X_{256}\\ X_{256}\\ X_{258}\\ X_{259} \end{pmatrix} = \begin{pmatrix} \frac{351}{125440}\\ \frac{113}{627200}\\ \frac{4559}{78400}\\ \frac{6609}{39200}\\ -\frac{41553}{313600}\\ \frac{26321}{313600}\\ -\frac{123}{7840}\\ \frac{351}{78400}\\ \frac{11}{12800}\\ \frac{1}{12800} \end{pmatrix}, \begin{pmatrix} X_{260}\\ X_{261}\\ X_{262}\\ X_{263}\\ X_{264}\\ X_{265}\\ X_{266}\\ X_{267}\\ X_{268}\\ X_{269} \end{pmatrix} = \begin{pmatrix} \frac{23}{14175}\\ \frac{13}{99225}\\ \frac{5494}{99225}\\ \frac{17632}{99225}\\ \frac{2174}{19845}\\ \frac{17632}{99225}\\ -\frac{5494}{99225}\\ \frac{32}{14175}\\ \frac{13}{99225}\\ 0 \end{pmatrix}, \begin{pmatrix} X_{270}\\ X_{271}\\ X_{272}\\ X_{273}\\ X_{274}\\ X_{275}\\ X_{276}\\ X_{277}\\ X_{278}\\ X_{279} \end{pmatrix} = \begin{pmatrix} -\frac{6275}{2032128}\\ \frac{425}{2032128}\\ \frac{15095}{254016}\\ \frac{21025}{127008}\\ \frac{137225}{1016064}\\ \frac{137225}{1016064}\\ \frac{21025}{127008}\\ \frac{15095}{254016}\\ -\frac{6275}{2032128}\\ \frac{425}{2032128} \end{pmatrix}, \begin{pmatrix} X_{280}\\ X_{281}\\ X_{282}\\ X_{283}\\ X_{284}\\ X_{285}\\ X_{286}\\ X_{287}\\ X_{288}\\ X_{289} \end{pmatrix} = \begin{pmatrix} -\frac{9}{9800}\\ 0\\ \frac{241}{4900}\\ \frac{477}{2450}\\ \frac{387}{4900}\\ \frac{209}{980}\\ \frac{387}{4900}\\ \frac{477}{2450}\\ \frac{241}{4900}\\ -\frac{9}{9800} \end{pmatrix}, \begin{pmatrix} X_{290}\\ X_{291}\\ X_{292}\\ X_{293}\\ X_{294}\\ X_{285}\\ X_{296}\\ X_{297}\\ X_{298}\\ X_{299} \end{pmatrix} = \begin{pmatrix} -\frac{12859}{1036800}\\ \frac{1169}{1036800}\\ \frac{13231}{129600}\\ \frac{637}{12960}\\ \frac{178801}{518400}\\ -\frac{65807}{518400}\\ \frac{25529}{64800}\\ -\frac{3521}{129600}\\ \frac{48769}{207360}\\ \frac{513}{12800} \end{pmatrix}$$