Asian Journal of Applied Science and Technology (AJAST)
Volume 1, Issue 9, Pages 53-77, October 2017

Fibre Bundles: A Conceptual Discourse

Prof. J. P. Singh

Professor, Department of Management Studies, Indian Institute of Technology Roorkee, Roorkee 247667, India.

Article Received: 03 August 2017 *Article Accepted: 28 September 2017* *Article Published: 10 October 2017*

ABSTRACT

The applications of fibre bundles are growing by the day. As of today, fibre bundles constitute the framework of choice for explaining general relativity and gauge theories in particle physics. Theories of economics and finance are also being premised on fibre bundles. This article attempts to provide a pedagogical introduction to fibre bundles in a manner that would enable the physicist to understand the structure purely at a conceptual level without getting entangled with extensive mathematical jargon.

Keywords: Manifolds, Fibre Bundles, Connections, Parallel Transport, Curvature and Covariant Derivatives.

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1. INTRODUCTION

Fibre bundles constitute the framework of choice for explaining general relativity and gauge theories in particle physics. Theories of economics and finance are also being premised on fibre bundles.

However, literature on fibre bundles can be segmented into (i) that which approaches the concept purely from the mathematical viewpoint, emphasizing the rigor rather than the philosophy underlying the concept and (ii) that which takes the acquaintance with fibre bundles for granted and get straight into the applications mode. The fallout of this segmentation is that both segments cover fibre bundles at the conceptual level only marginally. The coverage is, nowhere near adequate for the novice to get sufficiently familiarized with the intricacies of fibre bundles and hence, appreciate the nuances that have led to the origin and development of the concept and attempt to apply it to novel physical problems. This article attempts to fill this gap by introducing and explaining fibre bundles in a manner that would enable the reader to understand the structure purely at a conceptual level without getting entangled with extensive mathematical jargon to the extent possible.

We shall begin our development of the formalism of fibre bundles with the definition and explanation of topological manifolds and carry it through to encompass differentiable manifolds, differentiable maps, tangent bundles, principal and associated bundles, connections, curvature and parallel transport culminating in the definition of covariant derivatives (a cardinal structure in general relativity) highlighting all the relevant mathematical structures on the way to the extent they find application in contemporary physics. It is strongly emphasized that this is a pedagogical note aimed at addressing certain gaps in the existing literature and no claim to originality is made.

2. TOPOLOGICAL MANIFOLDS

2.1 Topological Manifolds: Physics is largely concerned with the study of systems that obey certain physical laws of motion. It becomes necessary to describe the positions of all the objects in the system in space-time by a set of numbers, or coordinates. These numbers (coordinates) may not be independent of each other and may instead satisfy some relations or constraints. These relations, put in the form of equations, can be interpreted to define a manifold in some Euclidean space. We, thus, define a topological manifold as: Let M is a topological space. Then, M is called a topological manifold of dimension n or a topological n-manifold if it has the following properties: (a) M is a Hausdorff space i.e. for every pair of points $p,q \in M$, there are disjoint open subsets $U,V \subset M$ such that $p \in U$ and $q \in V$; (b) M is second countable i.e. there exists a countable basis for the topology of M; (c) M is locally Euclidean of dimension n i.e. every point of M has a neighborhood that is homeomorphic to an open subset f of R^n. In other words, for each $p \in M$, we can find the following:(i) an open set $U \subset M$ containing p; (ii) an open set $\tilde{U} \subset R^n$; and (iii) a homeomorphism $\varphi : U \to \tilde{U}$.

Requiring a manifold to have the Hausdorff property ensures that one-point sets are closed and limits of convergent sequences are unique. Similarly, second countability has important consequences related to the existence of partitions of unity. Furthermore, both the Hausdorff and second countability properties are inherited by spaces that are built out of other manifolds e.g. subspaces and product spaces. It, therefore, follows that any open subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology). Similarly, any product of two manifolds is also a manifold equipped with the product topology.

2.2 Coordinate Charts & Atlases on Topological Manifolds: Let M be a topological n-manifold. A coordinate chart on M is a pair (U,φ), where U is an open subset of M and $\varphi : U \to \tilde{U} \subset R^n$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subset R^n$.

By definition of a topological manifold above, each point $p \in M$ is contained in the domain of some chart (U,φ). Furthermore, if $\varphi(p)=0$, we say that the chart is centered at p. The compatibility condition of two coordinate charts (U,φ) and (V,ψ) on their region of overlap $U \cap V$ if $U \cap V \neq \varnothing$ is expressed as the requirement of the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ (called the coordinate transition map from φ to ψ) being continuous (Figure 1).

$$
\begin{array}{c}
U \cap V \\
\varphi \swarrow \quad \searrow \psi \\
R^n \supseteq \varphi(U \cap V) \xrightarrow{\psi \circ \varphi^{-1}} \psi(U \cap V) \subseteq R^n
\end{array}
$$

Figure 1

For a given topological manifold M, the set of charts A such that $A = \bigcup_{(U,\varphi)\in A} U = M$ i.e. that comprise of charts covering the entire manifold M is called an atlas.

Given a chart (U,φ), the set U is called a coordinate domain, or a coordinate neighborhood of each of its points. If, additionally, $\varphi(U)$ is an open ball in R^n, then U is called a coordinate ball. The map φ is called a (local) coordinate map and the component functions $\left(x^1,...,x^n\right)$ of φ defined by $\varphi(p) = \left(x^1(p),...,x^n(p)\right)$ are called local coordinates of p on U.

Charts on topological manifolds are required only to verify that the set is a topological manifold as claimed. The chart is not part of the point-set definition. There is a largely unspoken assumption in the placing of topological charts on manifolds that the topology on the chart accurately reflects the native topological structure on the point set. In other words, if we define a continuous function or curve on the point set, then that function or curve will also be continuous within our mathematical model, and vice versa.

A topological manifold atlas is not just an atlas of charts which are compatible with each other. The charts must be compatible with the "native topology". And, if all charts are compatible with the native topology, they will consequently be compatible with each other. Thus compatibility of charts on their overlap (i.e. continuity of the transition maps) is a necessary condition, not a sufficient condition, for a valid atlas.

A topological space (X, Υ) requires no extra structure in addition to the topology Υ in order to be declared a topological manifold. Additional structure such as an atlas is optional. Differentiable manifolds do need extra structure (such as an atlas) for their specification. However, in practice, it is usually the coordinate patches which specify the topology because the topology is too difficult to define without patches. For example, the topology on $S^2 = \left\{x \in R^3; |x| = 1\right\}$ may be easily defined without charts. A metric function $d: S^2 \times S^2 \to R_0^+$ may be defined by $d(x,y) = |x-y|_{R^3} = \left(\sum_{i=1}^3 (x_i - y_i)^2\right)^{1/2}$ for all $s, y \in S^2$. This is easily shown to be a metric (particularly since it is the restriction to S^2 of the standard metric on R^3). Using projections onto the three axial planes, it is readily seen that the induced topological space is a topological manifold.

The important point to note from this example is that the topological space S^2 is a topological manifold without providing charts. The provision of charts only serves to verify that the topology meets the requirements.

But to meet the requirements of a differentiable manifold, an atlas of charts must be provided because a topological manifold has no specific differentiable structure unless an atlas is provided, and two atlases may very easily specify two incompatible differentiable structures on the same manifold.

3. Topological Bundles

3.1 Topological Bundles*:* A topological bundle is a triple (E,π,M) where E and M are topological manifolds called the total space and base space respectively and $\pi: E\to M$ is a surjective continuous map, called the projection map, from the total space E to the base space M. Let $p\in M$, then the pre-image $preim_\pi(\{p\}) = F_p$ is called the fibre at the point p of M.

The product manifold $M \times F$ of topological manifolds M and F can be interpreted as a topological bundle by setting $E = M \times F$ and the surjective continuous projection map $\pi: E\to M$ by $(m, f) \mapsto m$ where $m\in M$ and $f\in F$. The Mobius strip is another example of a topological bundle, but it is not a product manifold.

3.2 Topological Fibre Bundle*:* Let $E \xrightarrow{\pi} M$ be a topological bundle. Let $preim_\pi(\{p\}) = F$ for some topological manifold F and for all $p\in M$. Then, $E \xrightarrow{\pi} M$ is called a topological fibre bundle with typical fibre F.

3.3 Section of a topological fibre bundle*:* Let $E \xrightarrow{\pi} M$ be a topological bundle. A map $\sigma: M\to E$ is called a section of the bundle if $\pi \circ \sigma = id_M$. Thus, the map σ must necessarily map a given base point into the fibre at that point.

In the special case of a product topological manifold, say, $E = M \times F$, if we define the projection map $\pi: M \times F \to M$ as $\pi(m, f) = m$, then, the section map takes the form $\sigma: M \to M \times F$ defined by $p \mapsto (p, s(p))$ for $p\in M$ where $s: M\to F$ map. It is obvious that given $p\in M$, the map $\sigma: M\to E$ must necessarily map the point $p\in M$ to a point in F_p, the fibre at p and no other fibre.

3.4 Topological Sub-bundle and Restricted Bundle*: The* topological bundle $E' \xrightarrow{\pi'} M'$ is a sub-bundle of the topological bundle $E \xrightarrow{\pi} M$ if E' and M' are respectively submanifolds of E and M and $\pi' = \pi|_{E'}$ is the restriction of π to E'. Given a topological bundle $E \xrightarrow{\pi} M$ and a sub-manifold N of M, the bundle $preim_\pi(N) \xrightarrow{\pi|_{preim_\pi(N)}} N$ is a restriction of the bundle $E \xrightarrow{\pi} M$ to N.

3.5 Bundle Morphisms*:* Let the two topological bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be given along with the maps $u: E\to E'$ and $f: M\to M'$ such that the Figure 2 commutes i.e. $\pi' \circ u = f \circ \pi$. Then, (u,f) is called a bundle morphism.

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow \pi & & \downarrow \pi' \\ M & \xrightarrow{f} & M' \end{array}$$

Figure 2

3.6 Bundle Isomorphisms*:* Two topological bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are called isomorphic as bundles if there exists bundle morphisms (u,f) and (u^{-1}, f^{-1}) such that Figure 3 commutes.

$$E \underset{u^{-1}}{\overset{u}{\rightleftharpoons}} E'$$
$$\downarrow \pi \quad \downarrow \pi'$$
$$M \underset{f^{-1}}{\overset{f}{\rightleftharpoons}} M'$$

Figure 3

Such (u,f) are called bundle isomorphisms. These are the relevant structure preserving maps for topological bundles i.e. maps that preserve the structure of the fibres.

***3.7 Bundle Local Isomorphisms**:* A topological bundle $E \xrightarrow{\pi} M$ is called locally isomorphic (as a bundle) to another topological bundle $E' \xrightarrow{\pi'} M'$ if for every $p \in M$ there exists an open set U such that $p \in U$ and that the restricted bundle $preim_\pi(U) \xrightarrow{\pi|_{preim_\pi(U)}} U$ is isomorphic to $E' \xrightarrow{\pi'} M'$.

***3.8 Trivial and Locally Trivial Topological Bundles**:* A topological fibre bundle $E \xrightarrow{\pi} M$ is called a trivial topological bundle if it is isomorphic to the product bundle $M \times F \xrightarrow{\pi = proj_1} M$. Similarly, a topological bundle $E \xrightarrow{\pi} M$ is locally trivial if it is locally isomorphic to some product bundle $M \times F \xrightarrow{\pi = proj_1} M$. Triviality implies local triviality but not conversely. The Mobius strip is locally trivial but not trivial. Locally, any section of a bundle can be represented as a map from the base space to the fibre.

***3.9 Pullback of a topological bundle**:* The pullback of a topological bundle $E \xrightarrow{\pi} M$ is a topological bundle $E' \xrightarrow{\pi'} M'$ where: $E' := \{(m', e) \in M' \times E \,|\, \pi(e) = f(m')\}$; $\pi'(m', e) \mapsto m'$; $u(m', e) \mapsto e$; $f : M' \to M$ & $u : E' \to E$ constitute a bundle morphism.

4. Differentiable Manifolds

***4.1 Differentiable Manifolds**:* The definition of topological manifolds is sufficient for studying topological properties of manifolds, such as compactness, connectedness, simple connectedness, and the problem of classifying manifolds up to homeomorphism... However, in applications involving physics, we invariably need to be able to do calculus on such manifolds. For this purpose, we need to add extra structure to the manifold which would enable us to define the concept of differentiability of functions i.e. which functions on the manifold are "smooth" or differentiable and to what extent.

To see what additional structure on a topological manifold is appropriate for discerning which functions are differentiable, consider an arbitrary topological n-manifold M. Each point in M is in the domain of a coordinate map φ: $U \to \tilde{U} \subset R^n$. A possible definition of a smooth function on M would be to say that f: $M \to R$ is smooth if and only if the composite function $f \circ \varphi^{-1} : \tilde{U} \to R$ is smooth in the sense of ordinary calculus. But this will make sense only if this property is independent of the choice of coordinate chart. To guarantee this independence, one needs to restrict attention to "smooth charts". Since smoothness is not a homeomorphism invariant property, the way to do this is to consider the collection of all smooth charts as a new kind of structure on M. With this motivation in mind, we now describe the details of the construction.

***4.2 Chart Transition Map**:* Let M be a topological n-manifold. If (U,φ), (V,ψ) are two charts such that $U \cap V \neq \varnothing$, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the transition map from φ to ψ. It is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U,φ), and (V,ψ) are said to be smoothly compatible if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism (Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of R^n, smoothness of this map is to be interpreted as having continuous partial derivatives of all orders).

***4.3 Atlas**:* We define an atlas for M to be a collection of charts whose domains cover the whole of M. An atlas A is called a smooth atlas if any two charts in A are smoothly compatible with each other.

***4.4 Maximal Smooth Atlas**:* We have defined a "smooth structure" on M by giving a smooth atlas, and defined a function $f : M \to R$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth for each coordinate chart (U,φ) in the atlas. However, one issue remains unresolved. In general, there will be many possible choices of atlas that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definition:

A smooth atlas A on M is maximal if it is not contained in any strictly larger smooth atlas. This just means that any chart that is smoothly compatible with every chart in A is already in A. A smooth structure on a topological n-manifold M can, then, be defined as a maximal smooth atlas, A. A smooth manifold is a pair (M, A) where M is a topological manifold and A is a maximal smooth atlas on M. The following are two important properties of the maximal atlas in relation to a smooth manifold M: (a) Every smooth atlas for M is contained in a unique maximal smooth atlas; (b) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.

***4.5 Differentiable Functions and Maps**:* Although the terms "function" and "map" are technically synonymous, in studying smooth manifolds it is often convenient to make a slight distinction between them. We use the term "function" for a map whose range is R (a real-valued function) or R^k for some k >1 (a vector-valued function). The word "map" or "mapping" shall mean any type of map, such as a map between arbitrary manifolds. We begin by defining smooth real-valued and vector-valued functions.

If M is a smooth n-manifold, a function $f: M \to R^k$ is said to be smooth if for every $p \in M$, there exists a smooth chart (U,φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\tilde{U} = \varphi(U) \subset R^n$. The function $\bar{f}: \varphi(U) \to R^k$ defined by $\bar{f}(x) = f \circ \varphi^{-1}(x)$ is called the coordinate representation of f. By definition, f is smooth if and only if its coordinate representation is smooth in some smooth chart around each point of the domain. Smooth maps have smooth coordinate representations in every smooth chart.

The definition of smooth functions generalizes easily to maps between manifolds. Let M, N be smooth manifolds, and let $F: M \to N$ be any map. We say that F is a smooth map if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V,ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$ (Figure 4).

$$\begin{array}{ccccc} & M \supseteq U & \xrightarrow{F} & V \subseteq N & \\ & \downarrow \varphi & & \downarrow \psi & \\ R^{\dim M} \supseteq & \varphi(U) & \xrightarrow{\psi \circ F \circ \varphi^{-1}} & \psi(V) & \subseteq R^{\dim N} \end{array}$$

Figure 4

The previous definition of smoothness of real valued functions can be viewed as a special case of this one, by taking $N = V = R^k$ and $\psi \equiv id_{R^k}: R^k \to R^k$.

If $F: M \to N$ is a smooth map, and (U, φ) and (V, ψ) are any smooth charts for M and N respectively, we call $\bar{F} = \psi \circ F \circ \varphi^{-1}$ the coordinate representation of F with respect to the given coordinates. We establish below that the above definition of a smooth map is chart independent (Figure 5).

$$\begin{array}{ccccc} R^{\dim M} \supseteq & \tilde{\varphi}(U) & \xrightarrow{\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1}} & \tilde{\psi}(V) & \subseteq R^{\dim N} \\ & \uparrow \tilde{\varphi} & & \uparrow \tilde{\psi} & \\ \tilde{\psi} \circ \psi^{-1} & M \supseteq U & \xrightarrow{F} & V \subseteq N & \tilde{\psi} \circ \psi^{-1} \\ & \downarrow \varphi & & \downarrow \psi & \\ R^{\dim M} \supseteq & \varphi(U) & \xrightarrow{\psi \circ F \circ \varphi^{-1}} & \psi(V) & \subseteq R^{\dim N} \end{array}$$

Figure 5

For this purpose, we consider a map $F: M \to N$ that is smooth so that for every $p \in M$, there exist smooth charts (U,φ) containing p and (V,ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\bar{F} = \psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$ Let us, now, consider a different pair of smooth charts $(U, \tilde{\varphi})$ containing $p \in M$ and $(V, \tilde{\psi})$ containing $F(p)$ with $F(U) \subset V$ from the same atlas. We need to establish that the composition $\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1}$ is smooth i.e. that the map $\tilde{\psi} \circ (\psi^{-1} \circ \psi) \circ F \circ (\varphi^{-1} \circ \varphi) \circ \tilde{\varphi}^{-1}$ $= (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi}^{-1})$ is smooth. Now, the map $\psi \circ F \circ \varphi^{-1}$ is smooth by assumption whereas the maps $\tilde{\psi} \circ \psi^{-1}$ and $\varphi \circ \tilde{\varphi}^{-1}$ are both chart transition maps and hence, must also be smooth by definition of smooth atlases. It follows that the composition of these three maps will also be smooth. An important corollary to this chart independence proposition is that a manifold that is C^k smooth can only guarantee chart independence of smoothness of a function upto C^k and no higher. This is because the chart transition maps being only C^k may not preserve the smoothness of higher orders of the map $\psi \circ F \circ \varphi^{-1}$

4.6 Diffeomorphisms: A map $f: M \to N$ is a diffeomorphism if f is bijective and both f and f^{-1} are smooth maps. Diffeomorphisms are the isomorphisms or structure preserving maps of differentiable manifolds.

4.7 Derivations: Given algebras A,B we define a derivation as a linear map D: $A \to B$ which additionally satisfies the Leibnitz rule viz. $D[f,g]_A = [Df,g]_B + [f,Dg]_B$.

5. Tangent & Cotangent Spaces

5.1 Space of smooth functions on a manifold: Let M be a smooth manifold. The set of all smooth functions from M to the set of real numbers R is denoted by $C^\infty(M)$. It can be endowed with a vector space structure over R by defining on $C^\infty(M)$ pointwise addition (+) viz. $(f+g)(p) = f(p) + g(p)\ \forall f, g \in C^\infty(M), p \in M$ and scalar multiplication (.), $(\lambda.f)(p) = \lambda.f(p)\ \forall \lambda \in R, f \in C^\infty(M), p \in M$ where the LHS represent (+), (.) operations to be carried out in $C^\infty(M)$ and the right hand sides represent the usual sum and product operations of reals R.

5.2 The Directional Derivative: Let M be a smooth manifold and let $\gamma: R \to M$ be a smooth curve through a point $p \in M$. Without loss of generality, we may arrange for $p = \gamma(0)$. Then, the directional derivative operator at the point p along the curve γ is the linear map: $X_{\gamma,p}: C^\infty(M) \xrightarrow{\sim} R^1$ defined by its action on an arbitrary $f \in C^\infty(M)$ as $f \mapsto (f \circ \gamma)'(0) \in R$. $X_{\gamma,p}$ is also called the tangent vector to the curve γ at the point $p \in M$.

5.3 Tangent Space: The set defined by $T_pM = \{X_{\gamma,p} \mid \gamma \text{ is a smooth curve through } p \in M\}$ i.e. the set of tangent vectors at a point in a smooth manifold to smooth curves through that point can be given a vector space structure over R by identifying suitable pointwise "addition (■)" and "scalar multiplication (●)" operations as ■:$T_pM \times T_pM \to T_pM$ defined by $(X_{\gamma,p}$■$X_{\delta,p})(f) = X_{\gamma,p}(f) + X_{\delta,p}(f)$ and ●:$R \times T_pM \to T_pM$ defined by $(\lambda$●$X_{\gamma,p})(f) = \lambda.X_{\gamma,p}(f)$ $\forall X_{\gamma,p}, X_{\delta,p} \in T_pM$, $f \in C^\infty(M)$, $\forall \lambda \in R$ and $\forall p \in M$ where the (+) and (.) on the RHS are in R.

We show that T_pM is closed under the aforesaid operations. Let us take the case of (■). We need to establish that $(X_{\gamma,p}■X_{\delta,p}) \in T_pM$ i.e. that there exists a curve $\sigma : R \to M$ such that $(X_{\gamma,p}■X_{\delta,p})(f)$ can be put in the form $X_{\sigma,p} f = (f \circ \sigma)'(0)$. Let (U,φ) be a chart containing the given point p of M. Let $\sigma : (R \ni I) \to M$ by $\sigma(\theta) = \varphi^{-1} \circ (\varphi \circ \gamma(\theta) + \varphi \circ \delta(\theta) - \varphi(p))$. Then, $\sigma(0) = \varphi^{-1} \circ (\varphi \circ \gamma(0) + \varphi \circ \delta(0) - \varphi(p)) = \varphi^{-1} \circ (\varphi(p) + \varphi(p) - \varphi(p)) = p$. Now, $X_{\sigma,p} f = (f \circ \sigma)'(0)$

$$= \left(f \circ \varphi^{-1} \circ (\varphi \circ \gamma(\theta) + \varphi \circ \delta(\theta) - \varphi(p)) \right)'(0)$$

$$= \partial_a \left(f \circ \varphi^{-1} \right).(\varphi(p))$$

$$.\left(\varphi^a \circ \gamma(\theta) + \varphi^a \circ \delta(\theta) - \varphi^a(p) \right)'(0)$$

$$= \partial_a \left(f \circ \varphi^{-1} \right).(\varphi(p)).\partial_a (\varphi \circ \gamma)(\varphi(p))$$

$$+ \partial_a \left(f \circ \varphi^{-1} \right).(\varphi(p)).\partial_a (\varphi \circ \delta)(\varphi(p))$$

$$= \partial_a \left(f \circ \varphi^{-1} \circ \varphi \circ \gamma \right)(\varphi(p)) + \partial_a \left(f \circ \varphi^{-1} \circ \varphi \circ \delta \right)(\varphi(p))$$

$$= \left(f \circ \varphi^{-1} \circ \varphi \circ \gamma \right)'(0) + \left(f \circ \varphi^{-1} \circ \varphi \circ \delta \right)'(0)$$

$= (f \circ \gamma)'(0) + (f \circ \delta)'(0) = X_{\gamma,p}(f) + X_{\delta,p}(f)$ as required.

To establish closure of T_pM under (●), we define $\sigma : (R \ni I) \to M$ by $\sigma(\theta) = \varphi^{-1} \circ (\varphi \circ \gamma(\lambda\theta))$ for $\lambda \in R$ so that $\sigma(0) = \varphi^{-1} \circ (\varphi \circ \gamma(0)) = \varphi^{-1}(\varphi(p)) = p$. Also

$$X_{\sigma,p} f = (f \circ \sigma)'(0) = \left(f \circ \varphi^{-1} \circ (\varphi \circ \gamma(\lambda\theta)) \right)'(0)$$

$$= \partial_a \left(f \circ \varphi^{-1} \right).(\varphi(p)) \left(\varphi^a \circ \gamma(\lambda\theta) \right)'(0)$$

$$= \lambda.\partial_a \left(f \circ \varphi^{-1} \right).(\varphi(p)).\partial_a (\varphi \circ \gamma).(\varphi(p))$$

$$= \lambda.\partial_a \left(f \circ \varphi^{-1} \circ \varphi \circ \gamma \right).(\varphi(p)) = \lambda.\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma \right)'(0)$$

$$= \lambda.(f \circ \gamma)'(0) = \lambda X_{\gamma,p}(f).$$

We, now, prove an important result that the dimension of T_pM as a vector space is equal to the dimension of M as a topological manifold. We establish this proposition by explicitly constructing a vector space basis from a chart at an arbitrary point. For this purpose, consider a smooth manifold M of dimension m and let (U,x) be a chart centered at the given point p of M so that $x(p) = \mathbf{0}_{R^m}$. Let $\gamma_a : R \to M$, $a=1,2,...,m$ be m curves through the point p such that $(x^b \circ \gamma_a)(\lambda) = \delta^b_a \lambda$ and $\gamma_a(0) = p$. Let $e_a = X_{\gamma_a,p}$ be the tangent vector to the curve γ_a at p. Then, for $f \in C^\infty(M)$, we have $e_a f = X_{\gamma_a,p} f = (f \circ \gamma_a)'(0) = \left(f \circ x^{-1} \circ x \circ \gamma_a \right)'(0)$

$$= \partial_b \left(f \circ x^{-1} \right)\left(x(\gamma_a(0)) \right)\left(x^b \circ \gamma_a \right)'(0)$$

$$= \partial_b \left(f \circ x^{-1} \right)(x(p))\delta^b_a = \partial_a \left(f \circ x^{-1} \right)(x(p)) = \left(\frac{\partial}{\partial x^a} f \right)_p .$$

Thus, $e_a = \left(\frac{\partial}{\partial x^a} \right)_p$ are the tangent vectors to the chart induced curve γ_a at the point p in M. It may be noted that the map $\partial_a : C^\infty(R^m, R) \to C^\infty(R^m, R)$ is called the a^{th} partial derivative of the function f evaluated at $(x(p))$, $(\partial_a)_{x(p)} : C^\infty(R^m, R) \to R$ as $\left(\frac{\partial}{\partial x^a} \right)_p$.

Our next step is to prove that $\left\{ \left(\frac{\partial}{\partial x^a} \right)_p ; a = 1,2,...,m \right\}$ constitutes a generating system for T_pM i.e. that $X = X^a \left(\frac{\partial}{\partial x^a} \right)_p ; X^a \in R, a = 1,2,...,m$ with the Einstein's summation convention for any arbitrary $X \in T_pM$. To prove this, consider a smooth curve $\mu : R \to M$ through the given point p such that $\mu(0) = p$. Let (U,x) be a smooth chart containing the given point p of M. Then, the action of the tangent vector to the curve μ at p on a smooth function f is given by $X_{\mu,p} f = (f \circ \mu)'(0) = \left(f \circ x^{-1} \circ x \circ \mu \right)'(0)$

$$= \partial_b \left(f \circ x^{-1} \right)(x(p))\left(x^b \circ \mu \right)'(0) = \left(\frac{\partial}{\partial x^b} f \right)_p \left(x^b \circ \mu \right)'(0)$$

$= \left(x^b \circ \mu \right)'(0) \left(\frac{\partial}{\partial x^b} \right)_p f$. Thus, $\left\{ \left(\frac{\partial}{\partial x^a} \right)_p ; a = 1,2,...,m \right\}$ form a generating system for T_pM with the respective components $\left(x^b \circ \mu \right)'(0)$. We, finally establish that the vectors of this generating system are linearly independent. For this, we need to show that $\kappa^a \left(\frac{\partial}{\partial x^b} \right)_p f = 0 \Rightarrow \kappa^a = 0 \ \forall a = 1,2,...,m$ and for any smooth function f. Now, since (U,x) is a smooth chart, we have $x : U \to x(U) \subseteq R^m$ is a homeomorphism so that its component maps $x^b : U \to S \subseteq R$ are continuous (but not homeomorphic since they map to a space of different dimension). For $f : U \to R$ consider the diagram (Figure 6).

$$\begin{array}{ccc} U & \xrightarrow{f} & R \\ x \searrow & & \nearrow f \circ x^{-1} \\ & x(U) & \end{array}$$

Figure 6

The map f will be smooth if $f \circ x^{-1} : x(U) \to R$ is smooth. Letting $f = x^b$, we find that x^b will be smooth if $x^b \circ x^{-1}$ is smooth. But the action of the map $x^b \circ x^{-1}$ on the coordinate set of any point in the given chart (U,x) is $\left(x^b \circ x^{-1} \right)(a^1,...,a^m) = a^b$ whence $x^b \circ x^{-1} \equiv proj_b$ which is a smooth map so that x^b is smooth. Hence, we can mathematically consistently obtain $0 = \kappa^a \left(\frac{\partial}{\partial x^b} \right)_p x^b$ $= \kappa^a \partial_a \left(x^b \circ x^{-1} \right) x(p) = \kappa^a \delta^b_a \Rightarrow \kappa^a = 0$. It immediately follows that $\left\{ \left(\frac{\partial}{\partial x^a} \right)_p ; a = 1,2,...,m \right\}$ constitutes a vector space basis of T_pM. To reiterate, if $T_pM \ni X = X^a\left(\frac{\partial}{\partial x^a}\right)_p$, then the real numbers X^a are the components of the vector X of T_pM in the tangent space basis $\left(\frac{\partial}{\partial x^a}\right)_p$ induced from the chart (U,x) containing p. The chart transition maps may be nonlinear. However, a transition of basis in the tangent space is accomplished through a linear transformation.

5.4 Cotangent Space & Gradient Operator: The cotangent space at a point p on a smooth manifold M is the dual of the tangent space at that point i.e. $T_p^*M = \left(T_pM\right)^*$ where T_p^*M is the cotangent space at $p \in M$ and '*' denotes "dual" space. T_p^*M is a also a vector space in its own right. For finite dimensional manifolds, there exists a non-canonical vector space isomorphism between T_pM and T_p^*M. A non-canonical isomorphism is one that needs extra structure to be specified for defining the relevant isomorphic maps.

Similarly, we can also construct the vector space of all (r,s) tensors at $p \in M$ defined by $\mathbf{T}_s^r\left(T_pM\right) = \left\{t \middle| T_p^*M^{\times r} \times T_pM^{\times s} \xrightarrow{\sim} R\right\}$.

Let $f \in C^\infty(M)$ be arbitrary. Then, at every $p \in M$ we have a linear map $d_p : C^\infty(M) \xrightarrow{\sim} T_p^*M$, $f \mapsto d_pf$ defined, for any $X \in T_pM$ by $\left(d_pf\right)(X) = Xf$. d_p is called the gradient operator at $p \in M$ and $\left(d_pf\right)$ is the gradient of the function f at $p \in M$. $\left(d_pf\right)$ is a covector and not a vector as can be seen from the following: Consider X to be a tangent vector to the level set $N_c(f) = \left\{p \in M \middle| f(p) = c\right\}$ i.e. $X \in T_pN_c$. Then, by definition, we have $\left(d_pf\right)(X) = Xf = (f \circ \gamma)'(0) = 0$ since f is constant everywhere in N_c.

We, now, explicitly identify the basis of the cotangent space (called the chart induced covector basis) at a point $p \in M$ where M is a smooth manifold of finite dimension m. Let (U,x) be a smooth chart with $p \in U$. Then, $x : U \to x(U) \subseteq R^m$ is a homeomorphism so that its component maps $x^b : U \to S \subseteq R$ are continuous. Consider the gradients d_px^i, $i = 1,2,...,m \in T_p^*M$.

We have $d_px^a\left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\frac{\partial}{\partial x^b}\right)_p x^a = \partial_b\left(x^a \circ x^{-1}\right)\left(x(p)\right) = \delta_b^a$ where we have used the definitions of the gradient operator in the first step and the partial derivative operator in the next. This shows that the covectors d_px^i, $i = 1,2,...,m \in T_p^*M$ form a basis of the cotangent space and that this basis is orthonormal to the tangent space basis.

5.5 Push Forward and Pull Back of Maps: Let $\phi : M \to N$ be a smooth map between smooth manifolds M and N. Then, the push forward ϕ_* of the map ϕ at a point $p \in M$ is the linear map $\phi_{*,p} : T_pM \to T_{\phi(p)}N$; $X \mapsto \phi_{*,p}(X)$ defined by $\phi_{*,p}(X)f = X(f \circ \phi)$ where $f : N \to R$ is any smooth function, $X \in T_pM$ and $\phi_{*,p}(X)f \in T_{\phi(p)}N$. To verify the consistency of the definition, we note that $\phi : M \to N$, $f : N \to R$ are both smooth, so that $f \circ \phi : M \to R$ is also smooth i.e. $f \circ \phi \in C^\infty(M)$ and hence, $X \in T_pM$ can act on $f \circ \phi$. $\phi_{*,p}$ is also called the derivative of f at p and constitutes the only linear map that can be constructed with the given data.

The tangent vector $X_{\gamma,p}$ of a smooth curve $\gamma : R \to M$ at $p \in M$ with $\gamma(0) = p$ is pushed forward to the tangent vector of the smooth curve $\phi \circ \gamma$ at $\phi(p)$ i.e. $\phi_{*,p}\left(X_{\gamma,p}\right) = X_{(\phi\circ\gamma),\phi(p)}$

For an arbitrary $f \in C^\infty(N)$, we have $\phi_{*,p}\left(X_{\gamma,p}\right)f$
$= X_{\gamma,p}(f \circ \phi) = (f \circ \phi \circ \gamma)'(0) = \left(f \circ (\phi \circ \gamma)\right)'(0)$
$= X_{(\phi\circ\gamma),\phi(p)}f$ since $(\phi \circ \gamma)'(0) = \phi(p)$ because $\gamma(0) = p$.

We, now, define the pullback of a smooth map $\phi : M \to N$ between smooth manifolds M and N. Then, the pullback ϕ^* of the map ϕ at a point $\phi(p) \in N$ is the linear map $\phi_p^* : T_p^*M \leftarrow T_{\phi(p)}^*N$; $\omega \mapsto \phi_p^*(\omega)$ defined by $\omega\left(\phi_{*,p}(X)\right) = \phi_p^*(\omega)X$ where $\omega \in T_{\phi(p)}^*N$ is any covector in N, $X \in T_pM$ and $\phi_{*,p}(X) \in T_{\phi(p)}N$. To verify the consistency of the definition, we note that ϕ_p^* is an element of the cotangent space of M at the point $p \in M$. It will therefore act on the tangent vectors at that point ($X \in T_pM$).

5.6 Immersions and Embeddings: A smooth map $\phi : M \to R^n$ from a smooth manifold M is called an immersion of M into R^n if the derivative of ϕ viz. $\phi_{*,p} : T_pM \xrightarrow{\sim} T_{\phi(p)}R^n$ is injective at any $p \in M$ whereas a smooth map ϕ: $M \to N$ (M,N smooth manifolds) is called an embedding if (i) ϕ is an immersion and (ii) $\phi(M) \cong_{top} N$ which also implies that $\phi(M) \equiv_{diffeo} N$ since both M and N are assumed smooth.

6. Transformation Groups & Group Actions

6.1 Transformation Groups and Group Actions: A transformation group is an algebraic system consisting of two sets and two operations. The active set (G,σ) is a group. The passive set X has no operation of its own. There is a binary operation: $\mu : G \times X \to X$ called the "action" of the group G on X. Transformation groups may be (i) left transformation groups or (ii) right transformation groups. By default, a transformation group is a left transformation group.

6.2 Left Transformation Group: A (left) transformation group on a set X is a tuple (G,X,σ,μ) where (G,σ) is a group, and the map $\mu: G\times X\to X$ satisfies the following: (i) $\forall g_1,g_2\in G,\forall x\in X, \mu(\sigma(g_1,g_2),x)=\mu(g_1,\mu(g_2,x))$ (ii) $\forall x\in X,\ \mu(e,x)=x$, where e is the identity of G. The map μ may be referred to as the (left) action map or the (left) action of G on X. The value $\mu(g,x)$ of the action of $g\in G$ on $x\in X$ is abbreviated as gx or $g.x$ or $g\cdot x$ for a left transformation group G acting on X.

L_g^μ denotes the function $\{(x,\ \mu(g,x));(g,x)\in Dom(\mu)\}$ where μ is the left action map of a left transformation group (G,X,σ,μ), for each $g\in G$. L_g denotes the function L_g^μ when the left action map μ is implicit in the context.

The function L_g^μ is a well-defined function with domain and range equal to X for any left transformation group (G,X,σ,μ), for all $g\in G$. The function $L_g\equiv L_g^\mu$ may also be referred to as the "left action map" of the left transformation group. $L_g: X\to X$ satisfies $L_g(x)=\mu(g,x)=gx$ for all $g\in G$ and $x\in X$.

Each left action map is a bijection on the point space. For any fixed $g\in G$, the map $L_g: X\to X$ must be a bijection, because $L_g\left(L_{g^{-1}}(x)\right)=L_{g^{-1}}\left(L_g(x)\right)=x\ \forall x\in X$.

A left transformation group (G,X,σ,μ) is equivalent to a group homomorphism $\phi:(G,\sigma)\to Aut(X)$ where $Aut(X)$ is the group of bijections of X. The map ϕ is defined by $\phi: g\mapsto(x\mapsto\mu(g,x))$. This group homomorphism is called a realization of the group G.

6.3 Effective (Left) Transformation Group: An effective (left) transformation group is a left transformation group (G,X,σ,μ) such that: $\forall g\in G\setminus\{e\},\exists x\in X, gx\neq x$. (In other words, $L_g=L_e$ if and only if $g=e$). Such a left transformation group (G,X,σ,μ) is said to act effectively on X.

No two group elements produce the same action in an effective transformation group. If (G,X,σ,μ) is an effective left transformation group and $g, g'\in G$ are such that $L_g=L_{g'}$, then $g = g'$. In other words, no two different group elements produce the same action. That is, the group element is uniquely determined by its group action. Conversely, a left group action which is uniquely determined by the group element must be an effective left action.

Effective transformation groups are isomorphic to subgroups of the autobijection group. If the point set X of an effective left transformation group (G,X,σ,μ) is the empty set or a singleton, the only possible choice for G is the trivial group $\{e\}$. Since the set of all left transformations of a set X must be a subgroup of the group of all bijections from X to X, it is clear that the group of an effective left transformation group must be isomorphic to a subgroup of this group of all bijections from X to X.

The group operation of an effective left transformation group is uniquely determined by the action map. Let (G,X,σ,μ) be an effective left transformation group. Let $g_1, g_2\in G$. Then $\mu(g_1g_2,x)=\mu(g_1,\mu(g_2,x))$ for all $x\in X$. Suppose that there are two group elements g_3, $g_4\in G$ such that $\mu(g_3,x)=\mu(g_4,x)=\mu(g_1g_2,x)$ for all $x\in X$. Then for all $x\in X$, $\mu(g_3\,g_4^{-1},x)\ =\mu(g_3,\mu(g_4^{-1},x))\ \ =\mu(g_4,\mu(g_4^{-1},x))$ $=\mu(g_4g_4^{-1},x)=\mu(e,x)=x$. This implies that $g_3g_4^{-1}=e$ or $g_3=g_4$ since the group action is effective. So the group element $\sigma(g_1,g_2)=g_1g_2$ is uniquely determined by the action map μ.

Group elements may be identified with group actions if the action is effective. The above property implies that the group operation σ of an effective left transformation group (G,X,σ,μ) does not need to be specified because all of the information is in the action map. An effective transformation group is no more than the set of left transformations $L_g: X\to X$ of the set X. If the group action is not effective, then there are at least two group elements g, $g'\in G$ which specify the same action $L_g=L_{g'}: X\to X$. Any group which is explicitly constructed as a set of transformations of a set will automatically be effective. If a left transformation group is effective, the group elements g and the corresponding left translations L_g may be used interchangeably. There is no danger of real ambiguity in this.

One can map effective left transformation groups on X to subgroups of bijections on X. So all left transformation groups are essentially subgroups of the symmetric group on X. The elements of the group G may be thought of as mere "tags" or "labels" for bijections on X.

6.4 Free Action of a (left) transformation group on a set: A left transformation group (G,X,σ,μ) is said to act freely on the set X if $\forall g\in G\setminus\{e\},\forall x\in X,\mu(g,x)\neq x$. That is, the only group element with a fixed point is the identity e. A free left transformation group is a left transformation group (G,X,σ,μ) which acts freely on X.

In the special case that X is the empty set, the group G is completely arbitrary. So all left transformation groups except the trivial group $\{e\}$ act freely on the empty set, but are not effective.

Let (G, X, σ, μ) be a left transformation group. If $X \neq \varnothing$ and *G* acts freely on *X*, then (G, X, σ, μ) is an effective left transformation group. Let $X \neq \varnothing$ and assume that *G* acts freely on *X*. Let $g \in G/\{e_G\}$. Let $x \in X$. Then $\mu(g, x) \neq x$ by definition of free action. Therefore $\forall g \in G \setminus \{e\}$, $\exists x \in X$, $gx \neq x$. So *G* acts effectively on *X* by definition of effective action. Again, let (G, X, σ, μ) be a left transformation group. Then *G* acts freely on *X* if and only if $\forall y \in X, \forall g_1, g_2 \in G,\ (\mu(g_1, y) = \mu(g_2, y)) \Rightarrow (g_1 = g_2)$. (In other words, *G* acts freely on *X* if and only if the map $\mu_y : G \to X$ defined by $g \mapsto \mu(g, y)$ is injective for all $y \in X$). To establish this, let (G, X, σ, μ) be a left transformation group which acts freely on *X*. Let $y \in X$ and $g_1, g_2 \in G$ satisfy $\mu(g_1, y) = \mu(g_2, y)$. Then, $y = \mu(e, y)$ $= \mu(g_1^{-1}, \mu(g_1, y))$ $= \mu(g_1^{-1}, \mu(g_2, y))$ $= \mu(g_1^{-1} g_2, y)$ by definition of left transformation where *e* is the identity of *G*. Therefore $g_1^{-1} g_2 = e$ .So $g_1 = g_2$. Suppose that (G, X, σ, μ) satisfies $\forall y \in X, \forall g_1, g_2 \in G, (\mu(g_1, y) = \mu(g_2, y)) \Rightarrow$ $(g_1 = g_2)$. Let $g \in G$ and $y \in X$ satisfy $\mu(g, y) = y$. Then $\mu(g, y) = \mu(e, y)$. So $g = e$. Hence *G* acts freely on *X*.

6.5 Left Translation Group*:* The (left) transformation group of *G* acting on *G* by left translation, or the left translation group of *G*, is the left transformation group (G, X, σ, σ).

Let (G, σ) be a group. Define the action map $\mu : G \times G \to G$ by $\mu : (g_1, g_2) \mapsto \sigma(g_1, g_2)$. Then the tuple (G, G, σ, μ) $\equiv (G, G, \sigma, \sigma)$ is an effective, free left transformation group of *G*. This is established as follows: For a left transformation group, the action map $\mu : G \times X \to X$ must satisfy the associativity rule $\mu(\sigma(g_1, g_2), x) \equiv \mu(g_1, \mu(g_2, x))$ for all $g_1, g_2 \in G$ and $x \in X$. If the formula for μ i.e. $\mu : (g_1, g_2)$ $\mapsto \sigma(g_1, g_2)$ is substituted into this rule with $X = G$, it follows easily from the associativity of σ. The transformation group acts freely because *G* is a group. Since the action of *G* on *G* is free, it is necessarily effective because $G \neq \varnothing$.

6.6 Transitive Left Translation Group*:* A transitive (left) transformation group is a left transformation group (G, X, σ, μ) such that $\forall x, y \in X, \exists g \in G, \mu(g, x) = y$ (In other words, $\forall x \in X, \{\mu(g,\ x), g \in G\} = X$). A left transformation group (G, X, σ, μ) is said to act transitively on *X* if it is a transitive left transformation group. A left transformation group (G, X, σ, μ) acts transitively on a non-empty set *X* if and only if $\exists x \in X, \{\mu(g,\ x), g \in G\} = X$ To establish this, let (G, X, σ, μ) be a transitive left transformation group. Then $\forall x \in X, \{\mu(g,\ x), g \in G\} = X$ .So clearly $\exists x \in X, \{\mu(g,\ x), g \in G\} = X$ if *X* is non-empty. Now, let (G, X, σ, μ) be a left transformation group. Let $x \in X$ satisfy $\{\mu(g,\ x), g \in G\} = X$. Let $y \in X$. Then $\mu(g_y, x) = y$ for some $g_y \in G$. So $\mu(g_y^{-1}, y) = x$ because the left action map is a bijection. Therefore $\mu(g, x) =$ $\mu(g, \mu(g_y^{-1}, y))$ $= \mu(g g_y^{-1}, y)$ for all $g \in$G. In other words, $X =$ $\{\mu(g, x), g \in G\} = \{\mu(g g_y^{-1}, y), g \in G\} = \{\mu(g', y), g' \in G\}$ i.e. $\{\mu(g, y), g \in G\} = X$.

6.7 Orbits & Stabilizers*:* The orbit of a left transformation group (G, X, σ, μ) passing through the point $x \in X$ is the set $Gx = \{gx, g \in G\}$. A left transformation group (G, X, σ, μ) is transitive if and only if all of its orbits equal the whole point set *X*. One orbit equals *X* if and only if all orbits equal *X*.

A left transformation group (G, X, σ, μ) acts transitively on a non-empty set *X* if and only if *X* is the orbit of some element of *X*. This follows from the fact that a left transformation group (G, X, σ, μ) acts transitively on a non-empty set *X* if and only if $\exists x \in X, \{\mu(g,\ x), g \in G\} = X$.

The orbit space of a left transformation group (G, X, σ, μ) is the set $\{Gx, x \in X\}$ of orbits of (G, X, σ, μ) passing through points $x \in X$. X/G denotes the orbit space of a left transformation group (G, X, σ, μ).

The orbit space can be shown to be a partition of the passive set *X* by noting that the relation (R, X, X) defined by $R = \{(x_1, x_2) \in X \times X \mid \exists g \in G, x_1 = g x_2\}$ is an equivalence relation whose equivalence classes are of the form *Gx* for $x \in X$. To elaborate, let (G, X, σ, μ) be a left transformation group. Let $x_1, x_2 \in X$ be such that $Gx_1 \cap Gx_2 \neq \varnothing$ Then $g_1 x_1 = g_2 x_2$ for some g_1, $g_2 \in G$. So for any $g \in G$, $g x_1 = g(g_1^{-1}) g_1 x_1 = g(g_1^{-1}) g_2 x_2 \in G x_2$. Hence $Gx_1 \subseteq Gx_2$. Similarly $Gx_1 \supseteq Gx_2$. So $Gx_1 = Gx_2$ and it follows that X/G is a partition of *X*.

The stabilizer of a point $x \in X$ for a left transformation group (G, X, σ, μ) is the group (G_x, σ_x) with $G_x = \{g \in G \mid gx = x\}$ and $\sigma_x = \sigma|_{G_x \times G_x}$.

A left transformation group (G, X, σ, μ) acts freely on X if and only if $G_x = \{e\}$ for all $x \in X$ i.e. the group acts freely if and only if every stabilizer is a singleton, whereas the group acts transitively if and only if the orbit space is a singleton.

***6.8 Left Transformation Group Homomorphisms**:* A (left) transformation group homomorphism from a left transformation group $(G_1, X_1, \sigma_1, \mu_1)$ to a left transformation group $(G_2, X_2, \sigma_2, \mu_2)$ is a pair of maps (χ, φ) with $\chi : G_1 \to G_2$ and $\varphi : X_1 \to X_2$ such that (i) $\chi(\sigma_1(g, h)) = \sigma_2(\chi(g), \chi(h))$ for all $g,h \in G_1$ i.e. $\chi(gh) = \chi(g)\chi(h)$ for $g,h \in G_1$.(ii) $\varphi(\mu_1(g, x)) = \mu_2(\chi(g), \varphi(x))$ for all $g \in G_1$, $x \in X_1$ i.e. $\varphi(gx) = \chi(g)\varphi(x)$ for $g \in G_1, x \in X_1$. A (left) transformation group monomorphism from (G_1, X_1) to (G_2, X_2) is a left transformation group homomorphism is a pair of maps (χ, φ) such that $\chi : G_1 \to G_2$ and $\varphi : X_1 \to X_2$ are injective. A (left) transformation group epimorphism from (G_1, X_1) to (G_2, X_2) is a surjective left transformation group homomorphism (χ, φ) such that $\chi : G_1 \to G_2$ and $\varphi : X_1 \to X_2$ are surjective. A (left) transformation group isomorphism from (G_1, X_1) to (G_2, X_2) is a left transformation group homomorphism (χ, φ) such that $\chi : G_1 \to G_2$ and $\varphi : X_1 \to X_2$ are bijections. A (left) transformation group endomorphism of a left transformation group (G, X) is a left transformation group homomorphism (χ, φ) from (G, X) to (G, X) .A (left) transformation group automorphism of a left transformation group (G, X) is a left transformation group isomorphism (χ, φ))from (G, X) to (G, X).

An equivariant map is a special case of a transformation group homomorphism. An equivariant map between left transformation groups (G, X_1, σ, μ_1) and (G, X_2, σ, μ_2) is a map $\varsigma : X_1 \to X_2$ which satisfies $\forall g \in G, \forall x \in X_1$, $\varsigma(\mu_1(g, x)) = \mu_2(g, \varsigma(x))$ i.e $\varsigma(gx) = g\varsigma(x)$ for $g \in G$, $x \in X_1$.

***6.9 Right Transformation Groups**:* A right transformation group on a set X is a tuple (G, X, σ, μ) where (G, σ) is a group, and the map $\mu : X \times G \to X$ satisfies (i) $\forall g_1, g_2 \in G, \forall x \in X$, $\mu(x, \sigma(g_1, g_2)) = \mu(\mu(x, g_1), g_2)$ (ii) $\forall x \in X$, $\mu(x, e) = x$ where e is the identity of G. The map μ may be referred to as the (right) action map or the (right) action of G on X.

For a right transformation group $\mu(x, g)$ is generally denoted as xg or $x.g$ for $g \in G$ and $x \in X$. The associative condition (i) ensures that this does not result in ambiguity since $x(g_1 g_2) = (xg_1)g_2$ for all $g_1, g_2 \in G$ and $x \in X$. R_g^μ denotes the function $\{(x, \mu(x, g)), (x, g) \in Dom(\mu)\}$, where μ is the right action map of a right transformation group (G, X, σ, μ) for each $g \in G$. R_g denotes the function R_g^μ when the right action map μ is implicit in the context.

The function R_g^μ is a well-defined function with domain and range equal to X for any right transformation group (G, X, σ, μ), for all $g \in G$. The function $R_g \equiv R_g^\mu$ may also be referred to as the "right action map" of the right transformation group. The right action map $R_g : X \to X$ satisfies $R_g(x) = \mu(x, g) = xg$ for all $g \in G$ and $x \in X$.

The right action map $R_g : X \to X$ is a bijection for each $g \in G$ because $R_g(R_{g^{-1}}(x)) = R_g^{-1}(R_g(x)) = x$ for all $x \in X$.

Effective right transformation groups, free right transformation groups, right translation groups, transitive right transformation groups, orbits and stabilizers of right transformation groups and right transformation group homomorphisms are defined in analogous manner to equivalent concepts for left transformation groups.

7. Lie Groups & Lie Algebras

***7.1 Lie Groups**:* A Lie group $(G, \bullet)$ is (a) a group G with group operation $\bullet$ (b) a smooth manifold G i.e. the group manifold G is endowed with a compatible topology and a smooth maximal atlas (c) the manifold $G \times G$ is also a smooth manifold inheriting a smooth atlas from G together with the product topology and there exists (i) a smooth map $\mu : G \times G \to G$ defined by $(g_1, g_2) \mapsto g_1 \bullet g_2$ and (ii) a smooth map $i : G \to G$ defined by $g \mapsto g^{-1}$ for all $g_1, g_2, g \in G$.

***7.2 Left Translation**:* Let $(G, \bullet)$ be a Lie group. Then, for any $g \in G$, there is a map $l_g : G \to G$ defined by $h \mapsto l_g(h) = g \bullet h$ for all $h \in G$ called the left translation with respect to $g \in G$. Each l_g is an isomorphism for, we have, $l_g(h) = l_g(h') \Rightarrow g \bullet h = g \bullet h' \Rightarrow g^{-1} \bullet g \bullet h = g^{-1} \bullet g \bullet h'$ $\Rightarrow h = h'$ where the existence of g^{-1} for every $g \in G$ is guaranteed because G is a group. Hence, l_g is an injective map. Now, let $h \in G$ be arbitrary. Then, $l_g(g^{-1} \bullet h) = g \bullet g^{-1} \bullet h = h$ whence, for every image point $h \in G$, there exists a point $g^{-1} \bullet h$ in the domain G of the map l_g, thereby establishing surjectivity. l_g being bijective and smooth constitutes a diffeomorphism.

7.3 Lie Algebras: A Lie algebra $(L,\oplus,\odot,[\![.,.]\!])$ is a *k*-vector space $(L,\oplus,\odot)$ over the field *F* equipped with a Lie bracket that satisfies (i) bilinearity i.e. $[\![.,.]\!]: L\times L \xrightarrow{\sim} L$ (ii) anti-symmetry $[\![x,y]\!]=-[\![y,x]\!]$ (iii) Jacobi identity: $[\![x,[\![y,z]\!]]\!]+[\![[\![y,[\![z,x]\!]]\!]+[\![z,[\![x,y]\!]]\!]=0$.

8. Differentiable Bundles

8.1 Differentiable Bundles: A (differentiable) fibre bundle (E,π,M,F,G) consists of the following elements: (i) A differentiable manifold *E* called the total space (ii) A differentiable manifold *M* called the base space (iii) A differentiable manifold *F* called the fibre (or typical fibre) (iv) A surjection map $\pi: E\to M$ called the projection map. The inverse image $\pi^{-1}(p): F_p \approx F$ is called the fibre at point *p*∈*M* (v) A Lie group *G* called the structure group, which acts on *F* on the left (vi) A set of open covering $\{U_j\}$ of *M* with a diffeomorphism $\phi_i: U_i\times F\to\pi^{-1}(U_i)$ such that $\pi\circ\phi_i(p,f)=p$. The map ϕ_i is called the local trivialization since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i\times F$ (vii) If we write $\phi_i(p,f)=\phi_{i,p}(f)$, the map $\phi_{i,p}: F\to F_p$ is a diffeo -morphism. On $U_i\cap U_j\neq\varnothing$, we require that $t_{ij}(p)\equiv\phi_{i,p}^{-1}\circ\phi_{j,p}: F\to F$ be an element of *G*. Then ϕ_i,ϕ_j are related by smooth map $t_{ij}: U_i\cap U_j\to G$ $\phi_j(p,f)=\phi_i(p,t_{ij}(p)f)$ The maps t_{ij} are called the transition functions.

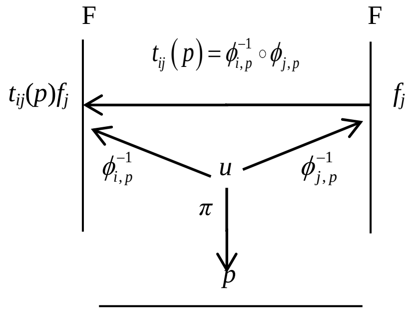
Strictly speaking, the definition of a fibre bundle should be independent of the special covering $\{U_j\}$ of *M*. In the mathematical literature, this definition is employed to define a coordinate bundle $(E,\pi,M,F,G,\{U_i\},\{\phi_i\})$.

Two coordinate bundles $(E,\pi,M,F,G,\{U_i\},\{\phi_i\})$ and $(E,\pi,M,F,G,\{V_i\},\{\psi_i\})$ are said to be equivalent if $(E,\pi,M,F,G,\{U_i\}\cup\{V_j\},\{\phi_i\}\cup\{\psi_j\})$ is again a co-ordinate bundle. A fibre bundle is defined as an equivalence class of coordinate bundles.

In practical applications in physics, however, we always employ a certain definite covering and make no distinction between a coordinate bundle and a fibre bundle.

Let us take a chart U_i of the base space *M*. $\pi^{-1}(U_i)$ is a direct product diffeomorphic to $U_i\times F$, $\phi_i^{-1}: \pi^{-1}(U_i)\to U_i\times F$ being the diffeomorphism. If $U_i\cap U_j\neq\varnothing$, we have two maps ϕ_i and ϕ_j on $U_i\cap U_j$. Let us take a point *u* such that $\pi(u)=p\in U_i\cap U_j$. We then assign two elements of *F*, one by ϕ_i^{-1} and the other by ϕ_j^{-1} i.e. $\phi_i^{-1}(u)=(p,f_i)$ and $\phi_j^{-1}(u)=(p,f_j)$. There exists a map $t_{ij}: U_i\cap U_j\to G$ which relates f_i and f_j as $f_i=t_{ij}(p)f_j$. This is also written as $\phi_j(p,f)=\phi_i(p,t_{ij}(p)f)$ (Refer figure 7).

We require that the transition functions satisfy the following consistency conditions: (i) $p\in U_i$ $t_{ii}(p)=id_F$ for (ii) $t_{ij}(p)=t_{ji}(p)^{-1}$ for $p\in U_i\cap U_j$ (iii) $t_{ij}(p)t_{jk}(p)=t_{ik}(p)$ for $p\in U_i\cap U_j\cap U_k$. Unless these conditions are satisfied, local pieces of a fibre bundle cannot be glued together consistently. If all the transition functions can be taken to be identity maps, the fibre bundle is called a trivial bundle. A trivial bundle is a direct product *M*×*F*.



$U_i\cap U_j$
Figure 7

Given a fibre bundle $E\xrightarrow{\pi}M$, the possible set of transition functions is not unique. Let $\{U_i\}$ be a covering of *M* and ϕ_i and χ_i be two sets of local trivializations giving rise to the same fibre bundle. The transition functions of the local trivializations are $t_{ij}(p)=\phi_{i,p}^{-1}\circ\phi_{j,p}$ and $\tilde{t}_{ij}(p)=\chi_{i,p}^{-1}\circ\chi_{j,p}$. Define a map $g(p): F\to F$ at each point *p*∈*M* by $g_i(p)\equiv\phi_{i,p}^{-1}\circ\chi_{i,p}$ .We require that $g_i(p)$ be a homeomorphism which belongs to *G*. This requirement must certainly be fulfilled if $\{\phi_i\}$ and $\{\chi_i\}$ describe the same fibre bundle. It is easily seen from $t_{ij}(p)=\phi_{i,p}^{-1}\circ\phi_{j,p}$, $\tilde{t}_{ij}(p)=\chi_{i,p}^{-1}\circ\chi_{j,p}$ and the definition $g_i(p)\equiv\phi_{i,p}^{-1}\circ\chi_{i,p}$ that $\tilde{t}_{ij}(p)=g_i(p)^{-1}\circ t_{ij}(p)\circ g_j(p)$.

In practical applications in physics, t_{ij} are the gauge transformations required for pasting local charts together, while g_i corresponds to the gauge degrees of freedom within a chart U_i. If the bundle is trivial, we may put all the transition functions to be identity. Then the most general form of the transition functions is $t_{ij}(p)=g_i(p)^{-1}g_j(p)$.

8.2 Sections & Local Sections: Let $E\xrightarrow{\pi}M$ be a fibre bundle. A section (or a cross section) $\sigma: M\to E$ is a smooth map which satisfies $\pi\circ\sigma=id_M$. Thus, $\sigma(p)=\sigma|_p$ is an element of $F_p=\pi^{-1}(p)$. The set of sections on *M* is denoted by Γ(*M*). If $U\subset M$, we can define a local section which is defined only on U. $\Gamma(U)$ denotes the set of local sections on U.

8.3 Reconstruction of fibre bundles: For given M, $\{U_i\}$, $t_{ij}(p)$, F and G, we can reconstruct the fibre bundle (E,π,M,F,G). This amounts to finding a unique π, E, ϕ_i from given data. For this purpose, we define, $X \equiv \bigcup_i U_i \times F$ and introduce an equivalence relation ~ between $(p,f) \in U_i \times F$ and $(q,f') \in U_j \times F$ by $(p,f) \sim (q,f')$ if and only if $p = q$ and $f' = t_{ij}(p) f$. A fibre bundle E is then defined as $X/\sim$. We denote an element of E by $[(p,f)]$. The projection is given by $\pi : [(p,f)] \mapsto p$. The local trivialization $\phi_i : U_j \times F \to \pi^{-1}(U_i)$ is given by $\phi_i : (p,f) \mapsto [(p,f)]$. π, E, ϕ_i thus defined satisfy all the axioms of fibre bundles. Thus, the given data reconstructs a fibre bundle E uniquely.

This procedure may be employed to construct a new fibre bundle from an old one. Let (E,π,M,F,G) be a fibre bundle. Associated with this bundle is a new bundle whose base space is M, transition function $t_{ij}(p)$, structure group G and fibre F' on which G acts.

8.4 Bundle Maps*:* Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be fibre bundles. A smooth map $u : E' \to E$ is called a bundle map if it maps each fibre F'_p of E' onto F_q of E. Then u naturally induces a smooth map $f : M' \to M$ such that $f(p) = q$ i.e. that the Figure 8 commutes i.e. $\pi \circ u = f \circ \pi'$.

$$\begin{matrix} E' & \overset{u}{\rightarrow} & E \\ \pi'\downarrow & & \downarrow\pi \\ M' & \overset{f}{\rightarrow} & M \end{matrix} \quad \left(\begin{matrix} w & \xrightarrow{u} & u(w) \\ \pi'\downarrow & & \downarrow\pi \\ p & \xrightarrow{f} & q \end{matrix} \right)$$

Figure 8

A smooth map $u : E' \to E$ is not necessarily a bundle map. It may map $x, y \in F'_p$ of E' to $u(x)$ and $u(y)$ on different fibres of E so that $\pi(u(x)) \neq \pi(u(y))$ in which case it is not a bundle map.

8.5 Equivalent bundles*:* Two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are equivalent if there exists a bundle map $u : E' \to E$ such that $f : M \to M$ is the identity map and u is a diffeomorphism.

8.6 Pullback bundles*:* Let $E \xrightarrow{\pi} M$ be a fibre bundle with typical fibre F. If a map $f : N \to M$ is given, the pair (E, f) defines a new fibre bundle over N with the same fibre F as shown in the Figure 9. Let f^*E be a subspace of $N \times E$ which consists of points (p,w) such that $f(p) = \pi(w)$. $f^*E \equiv \{(p,w) \in N \times E \mid f(p) = \pi(w)\}$ is called the pullback of E by f. The fibre F_p of f^*E is just a copy of the fibre $F_{f(p)}$ of E. If we define $f^*E \xrightarrow{\pi_1} N$ by $\pi_1 : (p,w) \mapsto p$ and $f^*E \xrightarrow{\pi_2} E$ by $(p,w) \mapsto w$, the pullback f^*E may be endowed with the structure of a fibre bundle and we obtain the following bundle map:

$$\begin{matrix} f^*E & \xrightarrow{\pi_2} & E \\ \pi_1\downarrow & & \downarrow\pi \\ N & \xrightarrow{f} & M \end{matrix} \quad \left(\begin{matrix} (p,w) & \xrightarrow{\pi_2} & w \\ \pi_1\downarrow & & \downarrow\pi \\ p & \xrightarrow{f} & f(p) \end{matrix} \right)$$

Figure 9

The commutativity of Figure 9 follows since $\pi(\pi_2(p, \mathrm{w})) = \pi(w) = f(p) = f(\pi_1(p, \mathrm{w}))$ for $(p,w) \in f^*E$. In particular, if $N = M$ and $f \equiv id_M$, then two fibre bundles f^*E and E are equivalent.

Let $\{U_i\}$ be a covering of M and $\{\phi_i\}$ be local trivializations. $\{f^{-1}(U_i)\}$ defines a covering of N such that f^*E is locally trivial. Take $w \in E$ such that $\pi(w) = f(p) \in U_i$ for some $p \in N$ If $\phi_i^{-1}(w) = (f(p), f_i)$ we find $\psi_i^{-1}(p,w) = (p, f_i)$ where ψ_i is the local trivialization of f^*E. The transition function t_{ij} at $f(p) \in U_i \cap U_j$ maps f_j to $f_i = t_{ij}(f(p)) f_j$. The corresponding transition function t^*_{ij} of f^*E at $p \in f^{-1}(U_i) \cap f^{-1}(U_j)$ also maps f_j to f_i. The shows that $t^*_{ij}(p) = t_{ij}(f(p))$.

9. Tangent Bundles

Let M be a smooth manifold of finite dimension m. Then the tangent bundle to M is the set $TM = \dot{\bigcup}_{p \in M} T_pM$ for all $p \in M$ i.e. the disjoint union of all the sets of all the tangent vectors at all the points of M with the bundle projection map $\pi : TM \to M$ defined by $T_pM \ni X_p \mapsto p \in M$ for all $X_p \in T_pM$. The set bundle so obtained can be converted to a smooth manifold bundle by constructing a smooth atlas on TM from a given smooth atlas on M as follows: Let A_M be a smooth atlas on M and let $(U,x) \in A_M$. From (U,x), we construct the chart $(preim_\pi(U), \chi)$. The set $preim_\pi(U)$ is, obviously, open in TM with the initial topology. We define the map $\chi : preim_\pi(U) \to \chi(preim_\pi(U)) \subseteq R^{2m}$ as $X \mapsto (x^1(\pi(X)), \ldots, x^m(\pi(X)), X^1, \ldots, X^m)$ for some $X \in TM$, i.e. X is an element of some tangent space, say T_pM. We note that since $\pi(X) \in U \subseteq M$ (π being the projection map, always maps X to an element of the base space M), the position of $\pi(X)$ in the base space M can be specified using the original chart (U,x) and is given by the set of m

coordinates $\left(x^1\left(\pi(X)\right),...,x^m\left(\pi(X)\right)\right)$. We still need to specify the location of *X* in the fibre. For this purpose, consider the basis of the tangent space (fibre) at the base point *π*(*X*) i.e. $T_{\pi(X)}M$ since *X* lies in this particular tangent space (fibre) over its own base point. This basis, in terms of the original chart is given by $\left\{\left(\frac{\partial}{\partial x^j}\right)_{\pi(X)}; j=1,2,...,m\right\}$. Since, this constitutes a basis of the tangent (fibre) space, we can expand *X*'s location in the fibre in terms of this basis as $X = X^j\left(\frac{\partial}{\partial x^j}\right)_{\pi(X)}; j=1,2,...,m$ in the tangent (fibre) space at *π*(*X*). It follows, then, that the set of *2m* coordinates $\left(x^i\left(\pi(X)\right), X^j; i,j=1,2,...,m\right)$ specify completely an element *X* of *TM*.

10. Vector Fields

10.1 Vector Fields: A vector field is simply a vector quantity that is a function of space-time. The difference between a vector and a vector field is that the former is one single vector while the latter is a distribution of vectors in space-time. The vector field exists in all points of space and at any moment of time. Therefore, while it suffices to expand any vector in a basis with real coefficients (because a single vector will have a set of constant coefficients in any given basis), it becomes necessary to use coefficients that are functions of space-time for the representation of vector fields or, in the alternative, use a space-time dependent basis. In case, we adopt the former approach, the components of vector fields in a given basis may be elements of $C^\infty(M)$. We, therefore, briefly touch upon the algebraic structure of $C^\infty(M)$.

$C^\infty(M)$ **as a vector space**: We can endow $C^\infty(M)$ with pointwise addition (+) and scalar multiplication (.) over R such that $\left(C^\infty(M),+,.\right)$ becomes a vector space over R. The operations (+) and (.) are defined in the usual manner as: $+: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$, $(f,g)(p)\mapsto(f+g)(p)$ $=(f)(p)+(g)(p)\ \forall f,g\in C^\infty(M),\ p\in M$ **&** $.: R\times C^\infty(M)$ $\to C^\infty(M)$ with $(\omega,f)(p)\mapsto(\omega.f)(p)=\omega(f(p))\ \forall f\in$ $C^\infty(M), \omega\in R, p\in M$.

$C^\infty(M)$ **as a ring**: We can endow $C^\infty(M)$ with pointwise addition (+) and multiplication (•) such that $\left(C^\infty(M),+,\bullet\right)$ becomes a ring (not a division ring). The operations (+) and (•) are defined as: $+: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$, (f,g) $(p)\mapsto(f+g)(p)=(f)(p)+(g)(p)\forall f,g\in C^\infty(M)$, $p\in M$ and $\bullet: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$, $(f,g)(p)\mapsto$ $(f\bullet g)(p)=(f)(p).(g)(p)\ \ \forall f,g\in C^\infty(M),\ p\in M$.

Consider, now, a smooth manifold *M* having the tangent bundle $TM\xrightarrow{\pi}M$ (*π* smooth). We define the vector field at a point $p\in M$ as the smooth section $\sigma(p)\in T_pM$. Thus, the map $\sigma: M\to TM$ satisfies $\pi\circ\sigma=id_M$. Let Γ be the set of all smooth vector fields on a smooth manifold *M* i.e. $\Gamma(TM)=\left\{\sigma: M\to TM \middle| \pi\circ\sigma(p)=id_M(p)\forall p\in M\right\}$. Then the set Γ(*TM*) can be endowed with a module structure over the ring $C^\infty(M)$ by defining the pointwise operations $(\oplus,\odot)$ in Γ(*TM*) as follows: $\oplus:\Gamma(TM)\times\Gamma(TM)\to$ $\Gamma(TM)$ $(\sigma,\tau)\mapsto\sigma\oplus\tau$ as $(\sigma\oplus\tau)(p)=\sigma(p)+\tau(p)$ $\forall\sigma,\tau\in\Gamma(TM), p\in M, +\in T_pM$ and $\odot: C^\infty(M)\times\Gamma(TM)\to$ $\Gamma(TM)$, $(f,\sigma)\mapsto f\odot\sigma$ as $(f\odot\sigma)(p)=f(p).\sigma(p)$ $\forall f\in C^\infty(M), \sigma\in\Gamma(TM), p\in M, .\in T_pM$. It can easily be verified that Γ(*TM*) is a module over the ring $C^\infty(M)$ under the aforesaid $(\oplus,\odot)$ operations. Furthermore, if the operation $\odot: C^\infty(M)\times\Gamma(TM)\to\Gamma(TM)$ is replaced by $\odot_F: F\times\Gamma(TM)\to\Gamma(TM)$, where F is any field e.g. the set of real numbers R or complex numbers $\mathbb{C}$, then Γ(*TM*) becomes a vector space over *F*.

It needs to be emphasized here that unlike a vector space, a module $(N,\oplus,\odot)$ over a ring *R* does not generically possess a basis, unless the ring *R* is a division ring. In fact, since a field is a division ring and a module over a field is a vector space, it follows that every vector space has a basis. However, there may exist modules over non-division rings that do have a basis. The important point is that such modules are not guaranteed to have a basis, although some such modules may possess a basis. Since Γ(*TM*) is a module over the ring $C^\infty(M)$ (which is not a division ring), it is not guaranteed that Γ(*TM*) will have a basis. Modules over a ring that possess a basis are called free modules while modules over rings that are direct summands of a free module over a ring are called projective. Thus, if a free module *N* over a ring *R* can be written as the direct sum *P*⨁*Q* where *Q* is another *R* module, then *P* is a projective *R* module. "Free" implies "projective" in the context of modules. Serre-Swan theorem explicitly establishes that the set of all smooth sections Γ(*E*) over a vector bundle $E\xrightarrow{\pi}M$ (where *M* is a smooth manifold) is a finitely generated projective $C^\infty(M)$ module i.e. Γ(*E*)⨁*Q*=*F* where *Q* is a $C^\infty(M)$ module and *F* is a free module. In a sense, *Q* quantifies the extent by which Γ(*E*) fails to have a basis.

It can also be shown that if *P* and *Q* are (finitely generated and/or projective) modules over commutative ring *R* then set $Hom_R(P,Q)=\left\{\varphi: P\xrightarrow{\sim}Q \middle| \varphi \text{ is linear}, \oplus,\odot \text{ pointwise}\right\}$ is again a (finitely generated and/or projective) module. In particular, we also have $Hom_{C^\infty(M)}$

$\left(\Gamma(TM), C^{\infty}(M)\right) = \Gamma(TM)^{*} = \Gamma\left(T^{*}M\right)$. This enables us to define a tensor field *t* on a smooth manifold *M* as a $C^{\infty}(M)$ multilinear map

$$t : \underbrace{\Gamma\left(T^{*}M\right)\times\ldots\times\Gamma\left(T^{*}M\right)}_{r\ copies}\times\underbrace{\Gamma(TM)\times\ldots\times\Gamma(TM)}_{s\ copies}\xrightarrow{\sim} C^{\infty}(M)$$

It is pertinent to recall that the pullback (defined for a covector at a point) can be extended to a covector field without additional conditions. However, the pushforward of a vector cannot be extended to the pushforward of a vector field unless the underlying smooth map is a diffeomorphism. To understand why this is so, we consider two smooth manifolds *M* and *N* and let $h : M \to N$ be a smooth map. Thus, each point of *M* is mapped to a unique point in *N* albeit two or more points of *M* may be mapped to the same point of *N* if *h* is not injective. Further, if *h* is not surjective, then there would be points in *N* that would not be the image of any point of *M* i.e. $h(M) \subset N$. Consider, now, a covector field defined over *N*. Then, every point of *N* would have a covector attached to it. The pullback of this covector field in *M* presents no problem because every point of *M* is associated with a point in *N* and hence, we can define a covector at every point of *M* i.e. a covector field over *M* by pulling back to every point in *M*, the covector in *N* to which that particular point in *M* maps under *h*. However, if we try to pushforward a vector field over *M* to *N*, we face problems on two counts. If *h* is not surjective, there would be points in *N* other than the image set *h*(*M*) on which vectors from *M* do not map under *h*. Hence, no field could be defined at these points by mapping of a vector field from *M* under *h*. At best, we could restrict the pushforward to the image set *h*(*M*) only. If the map *h* is not injective there could be two or more vectors of *M* mapping to the same point of *N* under *h*. In such a case, the vector field at the same image point of two or more vectors of *M* would not be well defined. Thus, in order that the pushforward of a vector field may exist, the underlying map must necessarily be bijective i.e. a diffeomorphism.

Now, since the left translation map $l_g : G \to G$ is a diffeomorphism, it can be used to pushforward any vector field *X* on *G* to another vector field defined by $\left(l_{g*}X\right)_{g,h\in G} = l_{g*}(X_h),\ X_h \in T_hG$.

***10.2 Left Invariant Vector Fields**:* Let *G* be a Lie group and *X* a vector field on *G*. Then, *X* is called a left invariant vector field if for any *g*∈*G*, $l_{g*}X = X$. This condition can also be expressed as $l_{g*}(X_h) = X_{gh},\ \forall g,h \in G$. The LHS gives, on action on an arbitrary $f \in C^{\infty}(M)$ (using the definition of pushforward) $\left(l_{g*}X_h\right)f = X_h\left(f \circ l_g\right) = \left[X\left(f \circ l_g\right)\right](h)$ and the RHS yields $X_{gh}f = (Xf)(gh) = \left[(Xf)\circ l_g\right](h)$ whence we can write the left invariant condition as $X\left(f \circ l_g\right) = (Xf)\circ l_g\ \ \forall f \in C^{\infty}(M), g \in G$. We denote the set of all left invariant vector fields on a Lie group *G* by $L(G)$. Obviously, then $L(G) \subset \Gamma(TG)$. We inherit $L(G)$ with the restrictions of the operations. $(\oplus,\odot)$ of Γ(*TG*) i.e. $\oplus : L(G)\times L(G) \to L(G)$ and $\odot : C^{\infty}(G)\ \times L(G) \to L(G)$. It can easily be verified that $L(G)$ is a module over the ring $C^{\infty}(G)$ under the aforesaid $(\oplus,\odot)$ operations. Like Γ(*TG*), if the scalar multiplication $\odot : C^{\infty}(G)\times L(G) \to L(G)$ is replaced by $\odot_F : F \times \Gamma(TG)\ \to \Gamma(TG)$, where *F* is any field, then $L(G)$ is a vector space over *F*.

The Lie algebra $\left(L(G), [\![.,.]\!]\right)$ is a sub-algebra of $\left(\Gamma(TG), [\![.,.]\!]\right)$ where $\Gamma(TG)$ is the space of all sections (vector fields) on the tangent bundle to the group *G*. By definition, we have $L(G) \subset \Gamma(TG)$. Further, the operations $\oplus,\odot$ are inherited from $\Gamma(TG)$. Hence, we only need to show that $[\![.,.]\!] : L(G)\times L(G) \xrightarrow{\sim} L(G)$. For this, we consider for $X, Y \in L(G)$, $[\![X,Y]\!]\left(f \circ l_g\right) = X\left(Y\left(f \circ l_g\right)\right) - Y\left(X\left(f \circ l_g\right)\right) = X\left((Yf)\circ l_g\right) - Y\left((Xf)\circ l_g\right) = \left(XY(f)\right)\circ l_g - \left(YX(f)\right)\circ l_g = \left([\![X,Y]\!]f\right)\circ l_g$ i.e. $[\![X,Y]\!]$ is a left invariant vector field in G, thereby confirming the subalgebraic structure of $L(G)$.

This Lie algebra $L(G)$ of the Lie group *G* is isomorphic as a vector space to the tangent space of *G* at the identity element i.e. $L(G) \cong_{vector\ space} T_eG$. To establish this, we consider the linear map $j : T_eG \xrightarrow{\sim} L(G),\ \ A \mapsto j(A)$ defined by $j(A)_g = l_{g*}A\ \forall g \in G$. Then, (i) $j(A)$ is left invariant i.e. $j(A) \in L(G)$ for $l_{h*}\left(j(A)_g\right)f = l_{h*}\left(l_{g*}A\right)f\ = \left(l_{h*}l_{g*}A\right)f = l_{g*}A\left(f \circ l_h\right) = A\left(f \circ l_h \circ l_g\right) = A\left(f \circ l_{hg}\right) = l_{(hg)*}f\ \ \forall f \in C^{\infty}(G)$ (ii) $j : T_eG \xrightarrow{\sim} L(G)$ is *R* linear (iii) $j : T_eG \xrightarrow{\sim} L(G)$ is injective for let $j(A) = j(B)$ for all *g*∈*G* $\Rightarrow j(A)_g = j(B)_g$ for all *g*∈*G* $\Rightarrow j(A)_e = j(B)_e \Rightarrow l_{e*}(A) = l_{e*}(B) \Rightarrow A = B$ (iv) $j : T_eG \xrightarrow{\sim} L(G)$ is also surjective for let $X \in L(G)$. We can show that there exists $A^X = X_e \in T_eG$ such that $j\left(A^X\right) = X$ for $j\left(A^X\right)_g = l_{g*}A^X = l_{g*}(X_e)\ = X_{ge} = X_g\ \Rightarrow j\left(A^X\right) = X \in L(G)$ (v) Finally, we identify the billinear map $[.,.] : T_eG\times T_eG \to T_eG$ in such a way that $\left(T_eG, [.,.]\right) \cong_{Lie\ algebra} \left(L(G), [\![.,.]\!]\right)$. This is done by setting $[.,.] : T_eG\times T_eG \to T_eG$ as $j\left([A,B]\right) = [\![j(A), j(B)]\!]$. The Lie algebra isomorphism between T_eG and $L(G)$ then takes the form $\psi : T_eG \to L(G)$ defined by $T_eG \ni [A,B] \mapsto j^{-1}[\![j(A), j(B)]\!] \in L(G)$.

11. VECTOR BUNDLES & FRAMES

***11.1 Vector Bundles**:* A vector bundle $E\xrightarrow{\pi}M$ is a fibre bundle whose fibre is a vector space. Let F be R^k vector space and M be an m-dimensional manifold. It is common to call k the fibre dimension and denote it by *dim E*, although the total space E is $(m+k)$ dimensional. The transition functions belong to $GL(k,R)$, since it maps a vector space onto another vector space of the same dimension isomorphically. If F is a complex vector space $\mathbb{C}^k$, the structure group is $GL(k,\mathbb{C})$.

A vector bundle whose fibre is one-dimensional ($F = R$ or $\mathbb{C}$) is called a line bundle. A cylinder $S^1\times R$ is a trivial R-line bundle. A Mobius strip is also a real line bundle. The structure group is $GL(1,R)=R-\{0\}$ or $GL(1,\mathbb{C})=\mathbb{C}-\{0\}$ which is abelian.

***11.2 Frames**:* On a tangent bundle *TM*, each fibre has a natural basis $\{\partial/\partial x^\mu\}$ given by the coordinate system x^μ on a chart U_i. We may also employ the orthonormal basis $\{\hat{e}_\alpha\}$ if M is endowed with a metric. $\partial/\partial x^\mu$ or $\hat{e}_\alpha$ is a vector field on U_i and the set $\{\partial/\partial x^\mu\}$ or $\{\hat{e}_\alpha\}$ forms linearly independent vector fields over U_i. It is always possible to choose m linearly independent tangent vectors over U_i but it is not necessarily the case throughout M. By definition, the components of the basis vectors are $\partial/\partial x^\mu = \left(0,\ ...,\ 0, \underset{\mu}{1}, 0, ...,\ 0\right)$ or $\hat{e}_\alpha=\left(0,\ ...,\ 0, \underset{\alpha}{1}, 0, ...,\ 0\right)$. These vectors define a (local) frame over U_i.

Let $E\xrightarrow{\pi}M$ be a vector bundle whose fibre is R^k (or $\mathbb{C}^k$). On a chart U_i, the piece $\pi^{-1}(U_i)$ is trivial, $\pi^{-1}(U_i)\cong U_i\times R^k$, and we may choose k linearly independent sections $\{e_1(p),...,e_k(p)\}$ over U_i. These sections are said to define a frame over U_i. Given a frame over U_i, we have a natural map $F_p\to F\left(=R^k \text{ or } \mathbb{C}^k\right)$ given by $V=V^\alpha e_\alpha(p)\mapsto \{V^\alpha\}\in F$. The local trivialization is $\phi_i^{-1}(V)=\left(p,\{V^\alpha(p)\}\right)$ By definition, we have $\phi_i\left(p,\left\{0,\ ...,0, \underset{\alpha}{1}, 0, ...,\ 0\right\}\right)=e_\alpha(p)$.

Let $U_i\cap U_j\neq\varnothing$ and consider the change of frames. We have a frame $\{e_1(p),...,e_k(p)\}$ on U_i and $\{\tilde{e}_1(p),...,\tilde{e}_k(p)\}$ on U_j where $p\in U_i\cap U_j$. A vector $\tilde{e}_\beta(p)$ is expressed as $\tilde{e}_\beta(p)=e_\alpha(p)G(p)^\alpha_{\ \beta}$ where $G(p)^\alpha_{\ \beta}\in GL(k,R)$ or $GL(k,\mathbb{C})$. Any vector $V\in\pi^{-1}(p)$ is expressed as $V = V^\alpha e_\alpha(p)=\tilde{V}^\alpha\tilde{e}_\alpha(p)$ whence we obtain the change of frames rule $\tilde{V}^\beta=G^{-1}(p)^\beta_{\ \alpha}V^\alpha$ where $G^{-1}(p)^\beta_{\ \alpha}G(p)^\alpha_{\ \gamma} = G(p)^\beta_{\ \alpha}G^{-1}(p)^\alpha_{\ \gamma}=\delta^\beta_\gamma$. Thus, we find that the transition function $t_{ij}(p)$ is given by a matrix $G^{-1}(p)$.

12. PRINCIPAL BUNDLES

Principal fibre bundles are immensely important in physics, since they constitute the contemporary apparatus for explaining general relativistic transformations as well as the Yang Mills theory in the standard model of particle physics. Very roughly speaking, principal fibre bundles are bundles whose fibres are Lie groups. They are immensely important objects since they enable us to understand any fibre bundle with a fibre on which a Lie algebra acts.

***12.1 Lie Group Action on a Manifold**:* Let $(G,\bullet)$ be a Lie group and let M be a smooth manifold. Since, G is also smooth, $G\times M$ is a smooth manifold and we can define a smooth map $\triangleright: G\times M\to M$ satisfying (i) $\exists e\in G|e\triangleright p = p,\ \forall p\in M$ and (ii) $g_2\triangleright(g_1\triangleright p)=(g_2\bullet g_1)\triangleright p$, $\forall g_1,g_1\in G$ and $p\in M$. The map $\triangleright: G\times M\to M$ is called the left G-action on M.

Similarly, we can define a right G-action on M as a smooth map $\triangleleft: M\times G\to M$ satisfying (i) $\exists e\in G|\ p\triangleleft e=p,\ \forall p\in M$ and (ii) $(p\triangleleft g_1)\triangleleft g_2=p\triangleleft(g_1\bullet g_2)$, $\forall g_1,g_1\in G$ and $p\in M$.

A correspondence between the left G-action and the right G-action can be defined as $p\triangleleft g=g^{-1}\triangleright p\ \ \forall g\in G$ and $p\in M$ for (i) given $\triangleleft$ as a smooth map, $\triangleright$ would also be smooth since the map $g\mapsto g^{-1}$ is smooth (ii) If $\exists e\in G|$ $p\triangleleft e=p,\ \forall p\in M$ then it follows that $p\triangleleft e=p= e\triangleright p,\ \forall p\in M$ i.e. $\exists e\in G|e\triangleright p=p,\ \forall p\in M$ and (iii) We have, $(p\triangleleft g_1)\triangleleft g_2 = (g_1^{-1}\triangleright p)\triangleleft g_2 = g_2^{-1}\triangleright(g_1^{-1}\triangleright p) = (g_2^{-1}\bullet g_1^{-1})\triangleright p=(g_1\bullet g_2)^{-1}\triangleright p=p\triangleleft(g_1\bullet g_2)\ \ \forall g_1,g_1\in G$ and $p\in M$ thereby establishing the correspondence.

***12.2 Equivariant Maps**:* Let $(G,\bullet)$, $(H,\circ)$ be two Lie groups with $\rho: G\to H$ being a (smooth) Lie group homomorphism i.e. $\rho(g_1\bullet g_2)=\rho(g_1)\circ\rho(g_2)\ \forall g_1,g_2\in G$ and let M and N be smooth manifolds. Let $\triangleright: G\times M\to M$ and $\triangleright': H\times N\to N$ be left G-action and left H-action defined on M and N respectively and let $f: M\to N$ be a smooth map. Then $f: M\to N$ is called ρ equivariant if $f(g\triangleright m)=\rho(g)\triangleright' f(m)\ \forall g\in G, m\in M$ i.e. if the Figure 10 commutes

$$\begin{array}{ccc} G\times M & \xrightarrow{\rho\times f} & H\times N \\ \triangleright\downarrow & & \downarrow\triangleright' \\ M & \xrightarrow{f} & N \end{array}$$

Figure 10

12.3 Orbit & Stabilizer: Define $\triangleright: G\times M \to M$ as the left *G*-action of a Lie group *G* on a smooth manifold *M*. (a) For any $p\in M$ we define its orbit under the left *G*-action $\triangleright: G\times M \to M$ as the set $\mathcal{O}_p = \{q\in M | \exists g\in G; g\triangleright p = q\}$ $\subseteq M$ i.e. the orbit of a point on a manifold is the set of all points that can be transformed to the given point by virtue of the given left action map. (b) Let us define a relation by $p\sim q \Leftrightarrow \exists g\in G; q = g\triangleright p$. It is easily seen that ~ is an equivalence relation so that we can constitute the set of equivalence classes as the quotient space $M/\sim = M/G$ which is called the orbit space of *M* under *G*. (c) Given $p\in M$, we define its stabilizer as the set S_p = $\{g\in G | g\triangleright p = p\} \subseteq G$ i.e. the stabilizer of a point on a manifold is the set of all group elements of *G* that leave the given point on the manifold invariant. (d) A left *G*-action $\triangleright: G\times M \to M$ is called free if $S_p = \{e\in G\}$, $\forall p\in M$. If $\triangleright: G\times M \to M$ is a free left *G*-action, then $\mathcal{O}_p \cong_{diffeomorphic} G$ for every $p\in M$. (e) The concepts of "orbit" and "stabilizer" can be defined in the context of a right *G*-action on a smooth manifold *M* similarly.

12.4 Principal G-Bundle: A smooth bundle $E \xrightarrow{\pi} M$ is called a principal *G*-bundle if (i) *E* is a right *G*-space i.e. *E* is equipped with a right *G*-action $\triangleleft: E\times G \to E$ (ii) this right action $\triangleleft: E\times G \to E$ is free so that every orbit of $\triangleleft$ is isomorphic to *G* (iii) the bundles $E \xrightarrow{\pi} M$ and $E \xrightarrow{\rho} E/G$ are isomorphic as bundles (where $E/G = E/\sim$ is the orbit space of *E* under *G* and $\rho: E \to E/\sim$ defined by $E \ni \varepsilon \mapsto [\varepsilon] \in E/\sim$ is the usual projection map from each element of *E* to its equivalence class under the right *G* action $\triangleleft$). Since the right action is free $\rho^{-1}([\varepsilon]) \cong_{diffeomorphism} G$ so that each fibre of $E \xrightarrow{\rho} E/G$ and hence, of $E \xrightarrow{\pi} M$ is diffeomorphic to the given Lie group.

An important example of principal bundle is the so-called frame bundle. Let *M* be a smooth manifold of dimension *m*. We define a frame at a point $x\in M$ as L_xM = $\{(e_1,...,e_m)_x | \text{such that } (e_1,...,e_m)_x$ constitute a basis of the tangent bundle T_xM }.i.e. it is the set of all basis sets to T_xM at a point $x\in M$. A frame bundle is, then, the disjoint union $LM = \dot{\bigcup}_{x\in M} L_xM$. We equip *LM* with a smooth atlas inherited from *M*. The projection map $LM \xrightarrow{\pi} M$ is defined by $LM \ni (e_1,...,e_m)_x \mapsto x\in M$ since there always exists a unique point *x* on the manifold *M* ($x\in M$) such that $(e_1,...,e_m)_x \in L_xM$ in *LM* at $x\in M$. In other words, $(e_1,...,e_m)_x \in L_xM$ can be projected onto the base manifold *M* by the map π since there exists a unique $x\in M$ corresponding to every $(e_1,...,e_m)_x \in L_xM \in LM$. With this projection map, $LM \xrightarrow{\pi} M$ is a bundle. We establish a right $GL(m,R) = \{g^m{}_n \in R, m,n = 1,2,...,m, \det(g)\neq 0\}$ action on *LM* by the following: $(e_1,...,e_m)\triangleleft (g\in GL(m,R)) = ((g^1{}_1e_1,...,g^1{}_me_1),...,(g^m{}_1e_m,...,g^m{}_me_m))$. It can easily be seen to be a free action because there does not exist any element $g\in GL(m,R)$ other than the group identity element that would leave the any given basis set invariant. We, finally, show that $LM \xrightarrow{\pi} M$ is a principal $GL(m,R)$ bundle. For this purpose, we need to prove that $LM \xrightarrow{\pi} M$ is isomorphic (as a bundle) to the bundle $LM \xrightarrow{\rho} LM/GL(m,R)$ where $LM/GL(m,R)$ is the orbit space of the right $GL(m,R)$-action $\triangleleft$ in *LM*. Let $p\in M$ be arbitrary. Then the frame at *p* is the element $L_pM \in LM$. Thus, the orbit of L_pM under the right $GL(m,R)$ action is the set of all frames $L_pM' \in LM$ at the same point $p\in M$ that can be obtained from L_pM by the action of an element of $GL(m,R)$. However, since the action is free, L_pM is invariant only under the action of identity *e* of $GL(m,R)$. The action of every other $g\in GL(m,R)$ will produce a distinct and different basis set but localized at the same $p\in M$ .But this new basis set will also be an element of $L_pM \in LM$ because this new basis set is also localized at *p* and $L_pM \in LM$ is the set of all the basis sets at *p*. Therefore, the equivalence class (orbit) of L_pM under the right $GL(m,R)$ action coincides with itself since L_pM is the set of all basis sets at $p\in M$ and the given right action produces another basis set at $p\in M$ i.e. another element of L_pM . Furthermore, all the basis sets at $p\in M$ i.e. all the elements of L_pM can be created from each other by the right action of the group $GL(m,R)$. Similar equivalence classes would exist corresponding to every other point of *M*. It follows that when the quotient set $LM/GL(m,R)$ is constructed by removing all the frames (created by the action of $GL(m,R)$ other than the identity element) at each point of *M*, we recover simply the manifold *M*, thereby establishing the isomorphism.

12.5 Principal Bundle Map: The pair of smooth maps (u, f) with $u: P\to P'$ and $f: M\to M'$ between two smooth principal bundles $(P,\pi,M,G,\triangleleft)$ and $(P',\pi',M',G',\triangleleft')$ constitute a principal bundle map if (i) there exists a Lie group homomorphism $\rho: (G,\bullet) \to (G',\bullet')$

defined by $\rho(g_1 \bullet g_2) = \rho(g_1) \bullet' \rho(g_2)\ \forall g_1, g_2 \in G$ (ii) $f \circ \pi = \pi' \circ u$ and (iii) $u(p \lhd g) = u(p) \lhd' \rho(g)\ \forall p \in P$, $g \in G$ i.e. that the Figure 11 commutes.

$$\begin{array}{ccccc} P & \xleftarrow{\lhd G} & P & \xrightarrow{\pi} & M \\ u\downarrow & & \downarrow u & & \downarrow f \\ P' & \xleftarrow{\lhd' G'} & P' & \xrightarrow{\pi'} & M' \end{array}$$

Figure 11

In Figure 11, if $f: M \to M'$ is the identity map i.e. $M = M'$ and further, $G=G'$, then the map $u: P \to P'$ is a diffeomorphism. (i) The injectivity of $u: P \to P'$ is proved as follows: Since $f: M \to M'$ is the identity map and $G = G'$ we have $\pi = \pi' \circ u$ and $u(p \lhd g) = u(p) \lhd' g\ \ \forall p \in P$, $g \in G$. Let $u(p_1) = u(p_2)$ for any $p_1, p_2 \in P$. Then, $\pi(p_1) = \pi'(u(p_1)) = \pi'(u(p_2)) = \pi(p_2)$ so that p_1, p_2 lie in the same fibre. Thus, there exists a unique $g \in G$ such that $p_1 = p_2 \lhd g$ since the action of G is free, as the given bundle is a principal bundle. Thus, $u(p_1) = u(p_2 \lhd g)$ $= u(p_2) \lhd' g = u(p_1) \lhd' g$. Since the action is free $g \in G$ must be the identity element of G whence $p_1 = p_2 \lhd g = p_2$. (ii) Let $p' \in P'$ be arbitrary. We need to look for some $p \in P$ such that $u(p) = p'$. We choose some point $p \in P$ in the fibre (under the projection map π) at the base point corresponding to the point $p' \in P'$ under the projection π' i.e. we choose $p \in \pi^{-1}(\pi'(p'))$ so that $\pi(p) = \pi'(p')$. This is possible because both π,π' map into the same manifold M. Then we have, $\pi'(u(p)) = \pi(p) = \pi'(p')$ so that $u(p)$ and p' belong to the same fibre. Since the action is free, there exists a unique $g \in G$ such that $u(p) \lhd' g = p'$ $\Rightarrow u(p \lhd g) = p' \Rightarrow u(\tilde{p}) = p'$. Thus, there exists $\tilde{p} \in P$ such that $u(\tilde{p}) = p'$.

12.6 Trivial Principal Bundles: A principal G- bundle $P \xleftarrow{\lhd G} P \xrightarrow{\pi} M$ is called trivial if it is diffeomorphic as a principal G-bundle to the principal G-bundle $M \times G \xleftarrow{\lhd' G} M \times G \xrightarrow{\pi_1} M$ with the G action $\lhd': (M \times G) \times G \to (M \times G)$ defined by $(x, g) \lhd' g' =$ $(x, g \bullet g')\ \forall x \in M, g, g' \in G$ and the projection map $\pi_1: M \times G \to M$, $\pi_1(x, g) = x\ \forall x \in M, g \in G$ or equivalently, if there exists principal bundle map $u: P \to M \times G$ in Figure 12.

$$\begin{array}{ccccc} P & \xleftarrow{\lhd G} & P & \xrightarrow{\pi} & M \\ u\downarrow & & \downarrow u & & \downarrow id \\ M \times G & \xleftarrow{\lhd' G} & M \times G & \xrightarrow{\pi_1'} & M \end{array}$$

Figure 12

We, now, establish the criterion for the triviality of a principal G-bundle and prove that a principal G-bundle is trivial if and only if there exists a smooth section $\sigma: M \to P$ i.e. a smooth map $\sigma: M \to P$ that satisfies $\pi \circ \sigma = id_M$. (i) Let us assume that the given principal bundle $P \xleftarrow{\lhd G} P \xrightarrow{\pi} M$ is trivial so that there exists a principal bundle map $u: P \to M \times G$. Then, we can construct a section $\sigma: M \to P$ at any $x \in M$ by the following prescription: $\sigma(x) = u^{-1}(x, id_G)$. The $\sigma: M \to P$ so defined is a section for $\pi(\sigma(x)) = \pi(u^{-1}(x, id_G)) = x$. (ii) Let $\sigma: M \to P$ be given such that $\pi \circ \sigma = id_M$ so that $\pi(\sigma(x)) = x\ \forall x \in M$. Consider an arbitrary $p \in P$. Then $\pi(p)$ is the projection of p to a point say, p_m on the base manifold M. Thus, $\sigma(\pi(p)) = \sigma(p_m)$ maps to a point in the fibre $\pi^{-1}(p_m)$ at the point p_m of the base manifold M. Since, all points of the same fibre are mapped to the same base point and since points in a fibre can be mapped to each other through the group G action, there exists a unique (unique, because the action of G is free) $\chi_\sigma(p) \in G$ such that $\sigma(\pi(p)) \lhd \chi_\sigma(p) = p$. Since $p \in P$ is arbitrary, the above equation holds for any other point $q = p \lhd (g \in G)$ of the same fibre so that we have, $\sigma(\pi(p \lhd g)) \lhd \chi_\sigma(p \lhd g) = p \lhd g\ \ \forall g \in G$. But since π projects all points of a fibre to the same base point, we have $\pi(p \lhd g) = \pi(p)$ so that the LHS gives $\sigma(\pi(p)) \lhd$ $\chi_\sigma(p \lhd g) = p \lhd g\ \ \forall g \in G$. Returning to the equation $\sigma(\pi(p)) \lhd \chi_\sigma(p) = p$ and taking the right action by an arbitrary element of G on both sides, we get $(\sigma(\pi(p)) \lhd \chi_\sigma(p)) \lhd g = \sigma(\pi(p)) \lhd (\chi_\sigma(p) \bullet g) = p \lhd g$ Since the right action is free, we obtain $\chi_\sigma(p \lhd g) = \chi_\sigma(p) \bullet g$. Let us, now, define the map $u_\sigma: P \to M \times G$ defined by $u_\sigma(p) = (\pi(p), \chi_\sigma(p))$. This is, obviously, a bundle map for $\pi_1(u_\sigma(p)) =$ $\pi_1(\pi(p), \chi_\sigma(p)) = \pi(p)\ \forall p \in P$. We, finally, establish that this is a principal bundle map. We have $u_\sigma(p \lhd g)$ $= u_\sigma(p) \lhd' g$. Also $u_\sigma(p \lhd g) = (\pi(p \lhd g), \chi_\sigma(p \lhd g))$ $= (\pi(p), \chi_\sigma(p) \bullet g) = (\pi(p), \chi_\sigma(p)) \lhd' g = u_\sigma(p) \lhd' g$, thereby confirming our assertion. Hence, the existence of a smooth section implies triviality of the principal G-bundle.

13. Associated Bundles

13.1 Associated Bundles: Given a principal G-bundle $P \xleftarrow{\lhd G} P \xrightarrow{\pi} M$ and a smooth manifold F with a left G-action defined by $\rhd: G \times F \to F$ by the same group G that acts on the principal G-bundle, we can define an associated bundle $P_F \xrightarrow{\pi_F} M$ where the projection map π_F and its domain P_F are as follows: (a) Set the relation $\sim_G$ on $P \times F$

by $P\times F \ni (p,f) \sim_G (p',f') \in P\times F \Leftrightarrow \exists g \in G$ such that $p' = p \lhd g$, $f' = g^{-1} \rhd f$. That $\sim_G$ is an equivalence relation is easily verified. We define P_F as the quotient space $P_F = (P\times F)/\sim_G$. In other words, the elements of P_F are the representative elements of the equivalence classes $[(p,f)], p\in P, f\in F$ where p, f are the representative elements from their orbits under the respective right G-action and left G-action. (b) The projection map $\pi_F : P_F \to M$ is defined by $P_F \ni [p,f] \mapsto \pi(p) \in M$. (c) to establish that $\pi_F : P_F \to M$ is well defined, we consider an arbitrary element of P_F. Obviously, this would be an element of some equivalent class, say $[(p,f)], p\in P, f\in F$ and hence, could be expressible as $[(p\lhd g, g^{-1}\rhd f)]$. Now, $\pi_F ([(p\lhd g, g^{-1}\rhd f)]) = \pi(p\lhd g) = \pi(p) = \pi_F([(p,f)])$. It can be shown that the above construction viz. $P_F \xrightarrow{\pi_F} M$ is a fibre bundle in its own right.

We, now, present some illustrations of associated bundles in the context of physical applications.

(i) Consider the frame bundle LM as the principal bundle (P) over a smooth manifold M of dimension m with the right $GL(m,R)$ action given by $(e_1,\ldots,e_m) \lhd (g \in GL(m,R)) = ((g^1{}_1 e_1,,\ldots,g^1{}_m e_1),\ldots,(g^m{}_1 e_m,,\ldots,g^m{}_m e_m))$. The fibre is the smooth manifold $F \equiv R^{\mathrm{m}}$ and the left action $\rhd: GL(m,R)\times R^{\mathrm{m}} \to R^{\mathrm{m}}$ is defined by $(g^{-1}\rhd f)^a = (g^{-1})^a{}_b f^b$. Then, $LM_{R^{\mathrm{m}}} \xrightarrow{\pi_{R^{\mathrm{m}}}} M$ is an associated bundle. There exists a bundle isomorphism $u: LM_{R^{\mathrm{m}}} \to TM$ s shown in Figure 13

$$\begin{array}{ccc} LM_{R^{\mathrm{m}}} & \xrightarrow{u} & TM \\ \pi_{R^{\mathrm{m}}}\downarrow & & \downarrow \pi_{TM} \\ M & \xrightarrow{id} & M \end{array}$$

Figure 13

defined by $LM_{R^{\mathrm{m}}} \left(= LM\times R^{\mathrm{m}}/\sim_{GL(m,R)}\right) \ni [e,f] \mapsto f^a e_a \in TM$ where e_a, $a=1,2,\ldots,m$ constitute a basis of the tangent bundle TM to the manifold M. This is invertible for, given an arbitrary $X\in TM$, we can choose a set of basis vectors e_a, $a=1,2,\ldots,m$ in the tangent space T_xM such that $X = f^a e_a$, $a=1,2,\ldots,m, (e_1,\ldots,e_m) \in L_xM \in LM$. We can, then, define the map $u^{-1}: TM \to LM_{R^{\mathrm{m}}}$ by $X \xrightarrow{u^{-1}} [e,f]$ where $[e,f]$ is the equivalence class of $e \equiv (e_a, a=1,2,\ldots,m), f\equiv (f^a, a=1,2,\ldots,m)$. This map is, thus, independent of the choice of basis set in the given equivalence class.

(ii) Consider the frame bundle LM as the principal bundle (P) over a smooth manifold M of dimension m with the right $GL(m,R)$ action given by $(e_1,\ldots,e_m) \lhd (g\in GL(m,R)) = ((g^1{}_1 e_1,,\ldots,g^1{}_m e_1),\ldots,(g^m{}_1 e_m,,\ldots,g^m{}_m e_m))$. The fibre is the smooth manifold $F \equiv (R^{\mathrm{m}})^{\times p} \times (R^{\mathrm{m}*})^{\times q}$ and the left action $\rhd: GL(m,R)\times\left((R^{\mathrm{m}})^{\times p}\times(R^{\mathrm{m}*})^{\times q}\right) \to \left((R^{\mathrm{m}})^{\times p}\times(R^{\mathrm{m}*})^{\times q}\right)$ is defined by $(g^{-1}\rhd f)^{i_1,i_2,\ldots,i_p}{}_{j_1,j_2,\ldots,j_q} = (f)^{\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_p}{}_{\tilde{j}_1,\tilde{j}_2,\ldots,\tilde{j}_q} (g^{-1})^{i_1}{}_{\tilde{i}_1} \ldots (g^{-1})^{i_p}{}_{\tilde{i}_p} (g)^{\tilde{j}_1}{}_{j_1} (g)^{\tilde{j}_q}{}_{j_q}$. Then, $LM_F \xrightarrow{\pi_F} M$ is an associated bundle isomorphic as a bundle to $T^p{}_q M \xrightarrow{\pi} M$ i.e. the (p,q) tensor bundle.

(iii) We define the principal bundle and fibre as in (ii) above but modify the left action as $(g^{-1}\rhd f)^{i_1,i_2,\ldots,i_p}{}_{j_1,j_2,\ldots,j_q} = (\det g)^{\omega} (f)^{\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_p}{}_{\tilde{j}_1,\tilde{j}_2,\ldots,\tilde{j}_q} (g^{-1})^{i_1}{}_{\tilde{i}_1} \cdots (g^{-1})^{i_p}{}_{\tilde{i}_p} (g)^{\tilde{j}_1}{}_{j_1} (g)^{\tilde{j}_q}{}_{j_q}$ for some $\omega\in\mathbb{Z}$. Then, the associated bundle $LM_F \xrightarrow{\pi_F} M$ is the (p,q)tensor density bundle of weight $\omega\in\mathbb{Z}$ over M.

(iv) We define the principal bundle as in (iii) and the fibre $F\equiv R$ and modify the left action as $(g^{-1}\rhd f) = (\det g).f$ leading to the definition of scalar density as an associated bundle $LM_F \xrightarrow{\pi_F} M$.

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***13.2 Associated Bundle Map**:* An ordered pair of maps $(\tilde{u},\tilde{h})$ between two associated bundles sharing the same fibre but being associated to arbitrarily different respective principal G-bundles is a bundle map which can be constructed from a principal bundle map (u,h) between the underlying principal bundles P and P' (i.e. $\pi'\circ u = h\circ\pi$, $u(p\lhd g) = u(p)\lhd' g$ (Figure 14)_

$$\begin{array}{ccccc} P & \xleftarrow{\lhd G} & P & \xrightarrow{\pi} & M \\ u\downarrow & & \downarrow u & & \downarrow h \\ P' & \xleftarrow{\lhd' G} & P' & \xrightarrow{\pi'} & M' \end{array}$$

Figure 14

such that Figure 15 commutes i.e. that $\tilde{u}: P_F \to P'_F$, $\tilde{u}([p,f]) = [u(p),f]$, $\tilde{h}: M\to M'$ $\tilde{h}(m) = h(m)$ $\forall p\in P, f\in F, m\in M$

$$\begin{array}{ccc} P_F & \xrightarrow{\tilde{u}} & P' \\ \pi_F\downarrow & & \downarrow \tilde{\pi}_F \\ M & \xrightarrow{\tilde{h}} & M' \end{array}$$

Figure 15

It is easily seen that two *F*- fibre bundles may be isomorphic as bundles but may, at the same time, fail to be isomorphic as associated bundles.

***13.3 Trivial Associated Bundles**:* An associated bundle is trivial if the underlying principal bundle is trivial. A trivial associated fibre bundle is always a trivial fibre bundle but the converse need not hold.

The sections $\sigma: M \to P$ of an associated fibre bundle $P_F \xrightarrow{\pi_F} M$ are in one to one correspondence to *F*-valued functions $\phi: P \to F$ on the underlying principal bundle. This can be established explicitly by constructing the section $s_\phi: M \to P_F$ corresponding to the given function $\phi: P \to F$ by $s_\phi(x) = [p, \phi(p)]\ \forall x \in M$ where $p \in \pi^{-1}(x)$. This $s_\phi: M \to P_F$ is well defined and invertible.

***13.4 Extension and Restrictions of Associated Bundles**:* Let *H* be a closed subgroup of the Lie group *G*. Let *P* be a principal *G*-bundle and *P*' be a principal *H*-bundle, both over the same base space *M*. If there exists a bundle map (u, f) such that /Figure 16 commutes

$$\begin{array}{ccccc} P & \xleftarrow{\triangleleft G} & P & \xrightarrow{\pi} & M \\ u\downarrow & & \downarrow u & & \downarrow f \\ P' & \xleftarrow{\triangleleft' H} & P' & \xrightarrow{\pi'} & M \end{array}$$

Figure 16

then $P \xrightarrow{\pi} M$ is called a *G*-extension of the principal *H*-bundle $P' \xrightarrow{\pi'} M$ and $P' \xrightarrow{\pi'} M$ is called the *H*-restriction of the principal *G*-bundle $P \xrightarrow{\pi} M$. It can be shown that any principal *H*-bundle can be extended to a principal *G*-bundle if *H* is a closed subgroup of *G*. A principal *G*-bundle can be restricted to a principal *H*-bundle if and only if (i) *H* is a closed subgroup of *G* and (ii) $P/H \xrightarrow{\pi'} M$ has a smooth section.

14. Connections

***14.1 Homomorphisms of Vector Fields and Lie Algebras**:* Let $P \xleftarrow{\triangleleft G} P \xrightarrow{\pi} M$ be a principal *G*-bundle. Then, each element *A* of the Lie algebra T_eG induces a vector field on the principal bundle *P* as: $X_p^A f = f(p \triangleleft \exp(tA))'(0) \forall p \in P$, $A \in T_eG,\ f \in C^\infty(P)$ where the prime (') denotes differentiation with respect to the parameter *t* and the differential is evaluated at *t* = *0*. We can, thus, define the Lie algebra homomorphism $i: T_eG \to \Gamma(TP)$, $A \mapsto X^A$ such that $i([A,B]_{T_eG}) = [i(A), i(B)]$ where $[.,.]_{T_eG}$ is the Lie bracket in T_eG.

***14.2 Vertical Subspaces**:* The vertical subspace $V_pP \subseteq T_pP$ at an arbitrary point $p \in P$ is defined by $T_pP \supseteq V_pP = \ker(\pi_*) = \{X \in T_pP | \pi_*(X) = 0\}$. It can be shown that $\forall p \in P$, $X_p^A \in V_pP$ i.e. that the vector fields generated by elements of the Lie algebra T_eG are vertical subspaces of the tangent space at a point $p \in P$. We have, $X_p^A f = f(p \triangleleft \exp(tA))'(0)\ \forall p \in P, A \in T_eG,\ f \in C^\infty(P)$. Now, the curve $\gamma(t) \equiv p \triangleleft \exp(tA)$ lies entirely within the fibre $\pi^{-1}(\pi(p))$. It follows that the push forward $\pi_*(X_p^A) = 0$ whence $X_p^A \in V_pP$.

***14.3 Connections on Principal Bundles**:* Let $P \xleftarrow{\triangleleft G} P \xrightarrow{\pi} M$ be a principal *G*-bundle. Then, a connection is an assignment where for every point $p \in P$, a vector subspace $H_pP \subseteq T_pP$ is chosen such that (a) $H_pP \oplus V_pP \equiv T_pP$ (b) $(\triangleleft g)_*(H_pP) = H_{p \triangleleft g}P$ (c) $T_pP \ni X_p = \underbrace{hor(X_p)}_{\in H_pP} + \underbrace{ver(X_p)}_{\in V_pP}$.(d) for every smooth vector field $X \in \Gamma(TP)$, both vector fields $hor(X)$, $ver(X)$ are smooth. It may be noted that both $hor(X_p)$, $ver(X_p)$ depend on the choice of H_pP.

***14.4 Connection One-form on Principal Bundles**:* The choice of a horizontal subspace H_pP at each point $p \in P$ of a principal *G*-bundle in order to provide a connection can be encoded in the thus induced Lie algebra valued one-form on *P* (not on the base manifold *M*) by the map $\omega_p: T_pP \xrightarrow{\sim} T_eG\ \forall p \in P$, $T_pP \ni X_p \mapsto \omega_p(X_p) \in T_eG$ where $\omega_p(X_p) = i_p^{-1}\left(\underbrace{ver(X_p)}_{\in V_pP \subseteq T_pP}\right) \in T_eG$ and $i_p: T_eG \to T_pP$, $A \mapsto X_p^A = ver(X_p)$. However, since X_p^A always lies in the vertical subspace V_pP, as proved above, the image of the map *i* restricts itself to V_pP and we can write $i_p: T_eG \to V_pP$. This map $i_p: T_eG \to V_pP$ is invertible because a Lie algebra always maps diffeomorphically to its Lie group and the fibre here is the Lie group *G*. $\omega_p(X_p)$ is called the connection one-form with respect to the given connection. We can recover the subspace H_pP from ω_p as $H_pP = \ker(\omega_p) = \{X \in T_pP | \omega_p(X) = 0\}$.

***14.5 Properties of Connection One-form**:* ω_p has the following important properties: (a) $\omega \circ i = id_{T_eG}$ for $\omega_p(i_p(A)) = \omega_p(X_p^A) = A = A \triangleleft id_{T_eG}$ because fields generated by Lie algebra elements of T_eG are vertical so that $ver(X_p^A) = X_p^A$ (b) $((\triangleleft g)^* \omega)(X_p) = (Ad_{g^{-1*}})(\omega_p(X_p))$. We observe that the left hand side is linear so that it suffices to prove the above separately for $X_p \in V_pP$ and $X_p \in H_pP$.

Let $X_p \in V_pP$. Then, there exists an element in the Lie algebra $A \in T_eG$ such that $X_p = X_p^A$. Then, $\left(\left(\triangleleft g\right)^* \omega\right)_p \left(X_p^A\right) =_{def\ of\ pullback} \omega_{p\triangleleft g}\left(\left(\triangleleft g\right)_* X_p^A\right) = \omega_{p\triangleleft g}$ $\left(X_{p\triangleleft g}^{Ad_{g^{-1}*}A}\right) = \omega_{p\triangleleft g}\left(X_{p\triangleleft g}^{Ad_{g^{-1}*}A}\right) = Ad_{g^{-1}*}A = Ad_{g^{-1}*}\left(\omega\left(X_p^A\right)\right)$.

Now, let $X_p = X_p^{hor} \in H_pP$. Then, $\left(\left(\triangleleft g\right)^* \omega\right)_p \left(X_p^{hor}\right)$ $=_{def\ of\ pullback} \omega_{p\triangleleft g}\left(\left(\triangleleft g\right)_* X_p^{hor}\right)$. But $\left(\triangleleft g\right)_* X_p^{hor} \in H_{p\triangleleft g}P$ (since the pushforward $\left(\triangleleft g\right)_* H_pP = H_{p\triangleleft g}P$. Hence, $\omega_{p\triangleleft g}$ $\left(\left(\triangleleft g\right)_* X_p^{hor}\right) = 0 = Ad_{g^{-1}*}\left(\omega\left(X_p^{hor}\right)\right)$ since $\left(\omega\left(X_p^{hor}\right)\right) = 0$.

(c) ω_p is smooth. This follows immediately from (a).

***14.6 Local Representation of Connection One-form**:* We recall that a connection ω was defined as a Lie algebra valued one-form i.e. $\omega : \Gamma(TP) \xrightarrow{\sim} T_eG$ satisfying (i) $\omega\left(X^A\right) = A$ and (ii) $\left(\left(\triangleleft g\right)^* \omega\right)(X) = Ad_{g^{-1}*}\left(\omega(X)\right)$. In practice, for computational purposes, it usually becomes necessary to restrict oneself to some open subset *U* of the base manifold *M*. We can, then, define the local representation of the connection ω in two ways over the two spaces viz. (i) $U \subseteq M$ (ii) $U \times G$. (i) We choose a local section $\sigma : U \to P$ defined by $\pi \circ \sigma = id_U$. Such a local section induces a Yang-Mills field on *U* viz. $\omega^U : \Gamma(TU) \to T_eG$ defined as $\omega^U = \sigma^* \omega$. This ω^U lives in *U* because of the pullback, (ii) We define the local trivialization of the principal *G*-bundle $h : U \times G \to P$ by $(m \in U, g \in G) \mapsto \sigma(m) \triangleleft g$. We can, then, define the local representation of ω by $\left(h^* u\right)_{(m,g)} : T_{(m,g)}(U \times G) \to T_mU$ $\oplus T_gG \in TP$. Thus, a pullback of ω to *U* yields the Yang Mills local representation on the base manifold and a pullback to $U \times G$ gives us the local representation on the local trivialization.

The relationship between these two representations is given by: $\left(h^*\omega\right)_{(m,g)}(v,\gamma) = Ad_{g^{-1}*}\left(\omega^U(v)\right) + \Xi_g(\gamma) \forall v \in T_mU, \gamma \in$ T_gG where $\Xi_g : T_gG \to T_eG$ with $L_g^A \mapsto A$ and L_g^A is the value of the left invariant vector field $l_{g*}A$ at $g \in G$ generated by the Lie algebra element $A \in T_eG$. $\Xi_g : T_gG \to T_eG$ is called the Maurer Cartan form on *G* and is a Lie algebra valued one form. We illustrate the above with reference to the frame bundle *LM* with *M* being a smooth manifold of finite dimension *d* and the structure group $G \equiv GL(d,R)$. We set up a coordinate chart (*U*,*x*) on the base manifold *M*. This chart induces a section at a point $m \in U$ given by $\sigma(m) = \left(\left(\frac{\partial}{\partial x^1}\right)_m, \ldots, \left(\frac{\partial}{\partial x^d}\right)_m\right)$. Then the Yang Mills field $\omega^U = \sigma^* \omega$ is a Lie algebra valued one-form on *U* with the components $\Gamma^i{}_{j,\mu} = \left(\omega^U\right)^i{}_{j,\mu}$, $i, j, \mu = 1,2,\ldots,d$. To calculate the Maurer-Cartan one-form on $GL(d,R)$ we identify an open subset $GL^+(d,R)$ of $GL(d,R)$ containing the group's identity element $id_{GL(d,R)}$ and set up the coordinate map $\mathbf{x} : GL(d,R) \to R^{dd}$ with $x^i{}_j$ $i, j = 1,2,\ldots,d : GL(d,R) \to R$ and $x^i{}_j(g) = g^i{}_j \forall g \in$ $GL(d,R)$. We,then, have $\left(L_g^A x^i{}_j\right) = \left(x^i{}_j\left(g \bullet \exp(tA)\right)\right)'(0)$ where the prime represents differentiation with respect to the parameter *t* and the derivative is evaluated at *t=0* and $\bullet$ is the group operation of $GL(d,R)$ .The expression $\left(g \bullet \exp(tA)\right) \equiv \gamma(t)$ constitutes a curve in *G* and $\gamma(0) = g$. Hence, the above equation represents the action of the vector field $X_{g,\gamma} \equiv L^A$ on the functions $f \equiv x^i{}_j \in C^\infty\left(GL^+\right)$ and evaluation the result at *g* of *G*. In view of the definition of the coordinates as $x^i{}_j(g) = g^i{}_j$, we can write the right hand side as $\left(g^i{}_k\left(\exp(tA)\right)^k{}_j\right)'(0) = g^i{}_k A^k{}_j$. Thus, we can write $L_g^A = g^i{}_k A^k{}_j \left(\frac{\partial}{\partial x^i{}_j}\right)_g$. The Maurer-Cartan form has to recover *A* from L_g^A so that it will be given by $\left(\Xi_g\right)^i{}_j = \left(g^{-1}\right)^i{}_k (dx)^k{}_j$ for $\left(\Xi_g\right)\left(L_g^A\right) = \left(g^{-1}\right)^i{}_k\left(dx^k{}_j\right)$ $\left(g^p{}_r A^r{}_q \frac{\partial}{\partial x^p{}_q}\right) = \left(g^{-1}\right)^i{}_k g^p{}_r A^r{}_q \delta^k{}_p \delta^q{}_j = \left(g^{-1}\right)^i{}_p g^p{}_r A^r{}_j = A^i{}_j$.

***14.7 Local Transition Rule for Connection One-form**:* Let *P* be a principal bundle with a smooth manifold *M* of finite dimension *d* as the base manifold. Let $\left(U^{(1)}, x\right)$ and $\left(U^{(2)}, y\right)$ be two overlapping charts on *M* so that $U^{(1)} \cap U^{(2)} \neq \varnothing$. Let $\sigma^{(1)} : U^{(1)} \to LM$ and $\sigma^{(2)} : U^{(2)} \to$ *LM* be sections from $U^{(1)}$ and $U^{(2)}$ respectively. We introduce a gauge transformation defined at every $m \in U^{(1)} \cap U^{(2)}$, $\Omega : U^{(1)} \cap U^{(2)} \to G$ by $\sigma^{(2)}(m) = \sigma^{(1)}(m) \triangleleft \Omega(m)$ which is possible because the right action of *G* in *P* is free. We, then, have the transformation rule $\omega_\mu^{(2)}(m) = Ad_{\Omega^{-1}(m)*}\omega_\mu^{(1)}(m) + \left(\Omega(m)^* \Xi_{m,\mu}\right)$. We illustrate this transformation rule by performing explicit calculations for the frame bundle *LM* as the principal bundle with the structure group $G \equiv GL(d,R)$. We set up two overlapping coordinate charts $\left(U^{(1)}, x\right)$ and $\left(U^{(2)}, y\right)$ as above and consider a point $p \in U^{(1)} \cap U^{(2)}$. We have $\Omega : U^{(1)} \cap U^{(2)}$ $\to G$ and $\Xi : T_gG \to T_eG$ so that $\Omega^*\Xi : T\left(U^{(1)} \cap U^{(2)}\right)$ $\to T_eG$. We take a an arbitrary basis vector $\left(\frac{\partial}{\partial x^\mu}\right)_p$ of

$T\left(U^{(1)}\cap U^{(2)}\right)$ at the point $p\in U^{(1)}\cap U^{(2)}$ and act on it by $\Omega^*\Xi$. We have $\left(\Omega^*\Xi\right)_p{}^i{}_j\left(\frac{\partial}{\partial x^\mu}\right)_p=\left(\Xi\right)_{\Omega(p)}{}^i{}_j\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)$ $=\left(\Omega(p)^{-1}\right)^i{}_k\left((dx)^k{}_j\right)_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)=\left(\Omega(p)^{-1}\right)^i{}_k$ $\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\left(x^k{}_j\right)=\left(\Omega(p)^{-1}\right)^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_p\left(x^k{}_j\circ\Omega\right)(p)$ $=\left(\Omega(p)^{-1}\right)^i{}_k\left(\frac{\partial}{\partial x^\mu}\right)_p\left(\Omega(p)\right)^k{}_j$ so that $\left(\Omega^*\Xi\right)_p{}^i{}_j=\left(\Omega(p)^{-1}\right)^i{}_k$ $\left(\frac{\partial}{\partial x^\mu}\right)_p\left(\Omega(p)\right)^k{}_j\,dx^\mu=\left(\mathbf{\Omega.d\Omega}\right)^i{}_j$. We, now, calculate $Ad_{\Omega^{-1}(p)*}\omega^{(1)}_\mu(p)$ explicitly. We note that the adjoint map is $Ad_g:G\to G$ such that $h\mapsto ghg^{-1}\ \forall h\in G$. Hence $Ad_g(e)=e$ whence $Ad_{g*}:T_eG\to T_{Ad_g(e)}G=T_eG$. It can be shown that given an arbitrary $A\in T_eG$, we have $Ad_{g*}A=\mathbf{gAg}^{-1}$ so that the transition rule between two Yang Mills fields on the same domain $U^{(1)}\cap U^{(2)}$ is $\left(\omega^{(2)}\right)^i{}_{j,\mu}=\left(\Omega^{-1}\right)^i{}_k\left(\omega^{(1)}\right)^k{}_{l,\mu}\Omega^l{}_j+\left(\Omega^{-1}\right)^i{}_k\partial_\mu\Omega^k{}_l$ Now, let, in the given case, $\sigma^{(1)}:U^{(1)}\to LM$ and $\sigma^{(2)}:U^{(2)}\to LM$ be sections from $U^{(1)}$ and $U^{(2)}$ respectively induced by the choice of the coordinate maps $x:U^{(1)}\to R^d$, $y:U^{(2)}\to R^d$ whence $\Omega^i{}_j=\frac{\partial y^i}{\partial x^j}$ and $\left(\Omega^{-1}\right)^i{}_j=\frac{\partial x^i}{\partial y^j}$ .are the Jacobians and $\left(\omega^{(2)}\right)^i{}_{j,\mu'}=\frac{\partial y^\mu}{\partial x^{\mu'}}\left[\frac{\partial x^i}{\partial y^k}\left(\omega^{(1)}\right)^k{}_{l,\mu}\frac{\partial y^l}{\partial x^j}+\frac{\partial x^i}{\partial y^k}\frac{\partial^2 y^k}{\partial x^l\partial x^\mu}\right]$ where $\mu\to\mu'$ relates to the transformation of coordinates that was not considered earlier under the premise that the connection one-form was kept independent of the coordinate transformations.

15. Parallel Transport

15.1 Horizontal Lift of a Curve: Let $\gamma:[0,1]\to M$ with $\gamma(0)=a$, $\gamma(1)=b$ be a smooth curve on the smooth base manifold *M*. Then the unique curve $\gamma^\uparrow:[a,b]\to P$ through a point $\gamma^\uparrow(0)=p\in\pi^{-1}(\{a\})$ which satisfies (i) $\pi\circ\gamma^\uparrow=\gamma$ (ii) $ver\left(X_{\gamma^\uparrow,\gamma^\uparrow(\lambda)}\right)=0\,\forall\lambda\in[0,1]$ (iii) $\pi_*\left(X_{\gamma^\uparrow,\gamma^\uparrow(\lambda)}\right)=X_{\gamma^\uparrow,\gamma^\uparrow(\lambda)}\,\forall\lambda\in[0,1]$ is called the horizontal lift of γ through *p*. There exist infinite number of horizontal lifts of a curve. However, specification of one point on the lift makes the lift unique.

15.2 Explicit Expression for Horizontal Lift of a Curve: We proceed in two steps :

(A) Generate the horizontal lift by starting from some arbitrary curve $\delta:[0,1]\to P$ that projects down to the original curve i.e. $\pi\circ\delta=\gamma$ by action of a suitable curve $g:[0,1]\to G$ such that $\gamma^\uparrow(\lambda)=\delta(\lambda)\triangleleft g(t)$. g will depend on the relationship between the position of δ and the corresponding point on the desired horizontal lift in each fibre.

(B) The curve $g:[0,1]\to G$ will be the solution to an ODE with the initial condition $g(0)=g_0$ where g_0 is the unique group element for which $\delta(0)\triangleleft g_0=p\in P$ through which the lifted curve is supposed to pass. We shall locally explicitly solve the ODE for $g:[0,1]\to G$ by a path ordered integral over the local Yang Mills field. The first order ODE for $g:[0,1]\to G$ is $Ad_{g(\lambda)^{-1}*}\left(\omega_{\delta(\lambda)}\left(X_{\delta,\delta(\lambda)}\right)\right)+\Xi_{g(\lambda)}\left(X_{g,g(\lambda)}\right)=0$ to be solved with the aforesaid initial condition. In the special case when *G* is a matrix group, this ODE takes the form $g(\lambda)^{-1}\bullet\omega_{\delta(\lambda)}\left(X_{\delta,\delta(\lambda)}\right)\bullet g(\lambda)+g(\lambda)^{-1}\bullet\dot{g}(\lambda)=0$ or $\omega_{\delta(\lambda)}\left(X_{\delta,\delta(\lambda)}\right)\bullet g(\lambda)+\dot{g}(\lambda)=0$ where the product ● denotes matrix multiplication and the superscript ● denotes differentiation with respect to *λ*. To solve this ODE locally, we focus on a chart (*U*,*x*)and choose a local section $\sigma:U\to P$ (so that $\pi\circ\sigma=id_U$). This induces two objects viz. (i) a curve $\delta=\sigma\circ\gamma$. This curve invariably projects down to γ on *M* i.e. $\pi\circ\delta=\gamma$ and (ii) Consider $X_{\gamma,\gamma(\lambda)}$ being an arbitrary tangent vector to the curve γ in *M*. The pushforward of $X_{\gamma,\gamma(\lambda)}$ under $\sigma:U\to P$ must yield $X_{\delta,\delta(\lambda)}$, the tangent vector to the curve $\delta=\sigma\circ\gamma$ i.e. $\sigma_*\left(X_{\gamma,\gamma(\lambda)}\right)=X_{\delta,\delta(\lambda)}$. Hence, we have $\omega_{\delta(\lambda)}\left(X_{\delta,\delta(\lambda)}\right)=\omega_{(\sigma\circ\gamma)(\lambda)}\left(\sigma_*X_{\gamma,\gamma(\lambda)}\right)=_{def\ of\ pullback}\left(\sigma^*\omega\right)_{\gamma(\lambda)}\left(X_{\gamma,\gamma(\lambda)}\right)$ $=_{def\ of\ Yang\ Mills\ fields}\Gamma^{U,\sigma}_{(\gamma(\lambda))}\left(X_{\gamma,\gamma(\lambda)}\right)=\Gamma^{U,\sigma}_{(\gamma(\lambda)),\mu}X^\mu{}_{\gamma,\gamma(\lambda)}$. Thus, the second object that is induced from the local section $\sigma:U\to P$ is the Yang Mills field $\Gamma^{U,\sigma}=\sigma^*\omega$. Making the substitution $\omega_{\delta(\lambda)}\left(X_{\delta,\delta(\lambda)}\right)=\Gamma^{U,\sigma}_{(\gamma(\lambda)),\mu}X^\mu{}_{\gamma,\gamma(\lambda)}$, dropping the indices *U*, *σ* and noting that $X_{\gamma,\gamma(\lambda)}$ is a tangent vector to γ and hence, can be expressed as the derivative $\dot{\gamma}(\lambda)$ we write the ODE in the special case of matrix groups as $\dot{g}(\lambda)=-\Gamma_{(\gamma(\lambda)),\mu}\dot{\gamma}(\lambda)^\mu g(\lambda)$ with the initial condition $g(0)=g_0$. This ODE can be integrated recursively to obtain the path ordered integral $g(t)=\left[\mathbf{P}\exp\left\{-\int_0^t d\lambda\Gamma(\lambda)\right\}\right]g_0$ where $\Gamma(\lambda)=\Gamma_{(\gamma(\lambda)),\mu}\dot{\gamma}(\lambda)^\mu$ and **P** denotes "path ordering". Thus, locally, the horizontal lift of the curve

Asian Journal of Applied Science and Technology (AJAST)
Volume 1, Issue 9, Pages 53-77, October 2017

$\gamma:[0,1]\to U \xrightarrow{\sigma} P$ is given by the explicit expression: $\gamma^{\uparrow}(\lambda)=(\sigma\circ\gamma)(\lambda)\lhd\left(\left[\mathbf{P}\exp\left\{-\int_0^{\lambda} dt\Gamma^{U,\sigma}_{\gamma(t),\mu}\dot{\gamma}(t)^{\mu}\right\}\right]g_0\right)$.

15.3 Parallel Transport*:* Let $\gamma:[0,1]\to M$ and let $\gamma^{\uparrow}:[0,1]\to P$ be its horizontal lift through $p\in\pi^{-1}(\{\gamma(0)\})$. Then, the parallel transport map $T_{\gamma}:\pi^{-1}(\{\gamma(0)\})\to\pi^{-1}(\{\gamma(1)\})$ is defined by $p\mapsto\gamma^{\uparrow}_p(1)$. This map is a bijection due to the fact that $(\lhd g)_* H_p = H_{p\lhd q}$.

Let $P\xleftarrow{\lhd G}P\xrightarrow{\pi}M$ be a principal *G*-bundle and let ω be a connection one-form on *P*. Let $P_F\xrightarrow{\pi_F}M$ be an associated bundle on whose typical fibre *F*, the group *G* acts from the left by its left action $\rhd$. We recall that $P_F=\{[(p,f)]\in P\times F\,|\,(p,f)\sim(p',f')\text{ iff }\exists g\in G; p'=p\lhd g, f'=g^{-1}\rhd f\}$. Let $\gamma:[0,1]\to M$ be a curve on the base manifold *M* and let $\gamma^{\uparrow}:[0,1]\to P$ be its horizontal lift through $p\in\pi^{-1}(\{\gamma(0)\})$. Then the horizontal lift of $\gamma:[0,1]\to M$ to the associated bundle that passes through $[(p,f)]\in P_F$ is the curve $\gamma^{\uparrow}_{P_F}:[0,1]\to P_F$ with $\gamma^{\uparrow}_{P_F}(\lambda)=[(\gamma^{\uparrow}(\lambda),f)]$. The parallel transport map is then defined in the associated bundle as $T^{P_F}_{\gamma}:\pi_F^{-1}(\{\gamma(0)\})\to\pi_F^{-1}(\{\gamma(1)\})$ defined by $[(p,f)]\to\gamma^{\uparrow P_F}_{[p,f]}(1)$. This map also provides a bijection between the fibres in the associated bundle.

16. Curvature & Torsion

16.1 Covariant Derivative*:* Let $P\xleftarrow{\lhd G}P\xrightarrow{\pi}M$ be a principal *G*-bundle and let ω be a connection one-form on *P*. Let Φ be a $\Diamond$ *k*- form, and define $D\Phi:\Gamma(T^k_0 P)\to\Diamond$ by $D\Phi(X_1,\dots,X_{k+1})=d\Phi(hor(X_1),\dots,hor(X_{k+1}))$. Then, *D* is called the covariant exterior derivative of Φ.

16.2 Curvature*:* Let $P\xleftarrow{\lhd G}P\xrightarrow{\pi}M$ be a principal *G*-bundle and let ω be a connection one-form on *P*. Then, the curvature of the connection one-form ω is the Lie algebra valued two-form on *P* given by $\Omega:\Gamma(T^2_0 P)\to T_eG$, $\Omega\equiv D\omega$. We have $\Omega=d\omega+\omega\wedge\omega$ where we define $\wedge$ by its action on a pair of vector fields on *P*, $X,Y\in\Gamma(TP)$ by $(\omega\wedge\omega)(X,Y)=[\![\omega(X),\omega(Y)]\!]$ where $[\![.,.]\!]$ is the Lie algebra bracket in T_eG. The compatibility of the above expression follows from the fact that on the LHS ω is Lie algebra valued while on the RHS, $[\![.,.]\!]$ is also Lie algebra valued. To establish that $\Omega=d\omega+\omega\wedge\omega$, we note that Ω is a two-form and hence, is billinear We can, therefore, establish this identity by proving it for the three cases (i) *X,Y* are both purely vertical (ii) *X,Y* are both horizontal and (iii) one (e.g. *X*) is horizontal and the other (*Y*) is vertical. (i) Let both *X,Y* be vertical so that both $X,Y\in\Gamma(TP)$ which means that they are generated by some elements of the Lie algebra T_eG i.e. $\exists A,B\in T_eG$ such that $X\equiv X^A$ and $Y\equiv X^B$.Hence, LHS is $\Omega(X^A,X^B)$ $=_{def\ of\ \Omega} D\omega(X^A,X^B)$ $=_{def\ of\ D} d\omega(hor(X^A),hor(X^B))=d\omega(0,0)=0$ since both $hor(X^A)$ $hor(X^B)$ vanish because *X,Y* are assumed vertical fields. The RHS gives $d\omega(X^A,X^B)+\omega\wedge\omega(X^A,X^B)=X^A(\omega(X^B))-X^B(\omega(X^A))-\omega([X^A,X^B])+[\![\omega(X^A),\omega(X^B)]\!]$. Now $\omega(X^{A,B})=_{def\ of\ \omega}A,B$. Further, there exists a Lie algebra homomorphism $T_eG\to\Gamma(TP)$ under which the generator of a field maps to the field i.e. $A,B\mapsto X^{A,B}$ and $\omega([X^A,X^B])=\omega(X^{[\![A,B]\!]})=[\![A,B]\!]=[\![\omega(X^A),\omega(X^B)]\!]$. Putting all these pieces together and using the commutativity of X^A and X^B, the RHS vanishes as well. (ii) Let both *X,Y* be horizontal. Then LHS is $\Omega(X,Y)=_{def\ of\ \Omega}D\omega(X,Y)$ $=_{def\ of\ D} d\omega(hor(X),hor(Y))=d\omega(X,Y)$ since both *X,Y* be horizontal so $hor(X)=X$, $hor(Y)=Y$. The RHS gives $d\omega(X,Y)+\omega\wedge\omega(X,Y)$ $=d\omega(X,Y)+[\![\omega(X),\omega(Y)]\!]$ $=d\omega(X,Y)$ because $\omega(X)=\omega(Y)_{def\ of\ \omega}=0$ since they have no vertical component. (iii) Without loss of generality, we may assume that *X* is horizontal and *Y* is vertical because Ω is a two form and hence antisymmetric so that if both LHS and RHS vanishes for *X* horizontal and *Y* vertical, they would also vanish for the opposite combination. Proceeding as in (i), we identify $Y=X^A$ for some $A\in T_eG$. Then LHS gives $\Omega(X,X^A)=d\omega(hor(X),hor(X^A))=0$ as $hor(X^A)=0$ as it has no horizontal component by assumption. RHS is $d\omega(X,X^A)+\omega\wedge\omega(X,X^A)=X(\omega(X^A))-X^A(\omega(X))-\omega([X,X^A])$ $+[\![\omega(X),\omega(X^A)]\!]$. Now $\omega(X)=_{def\ of\ \omega}0$, $\omega(X^A)=_{def\ of\ \omega}A$, . Also $\omega([X,X^A])=0$ since $[X,X^A]$ is horizontal. $X(A)=0$ as *X* is horizontal, by assumption, and *A* is the generator of a vertical vector field. Hence, the RHS vanishes as well, thereby establishing the desired equality.

If *G* is a matrix group, then we can write $\Omega^i{}_j=d\omega^i{}_j+\omega^i{}_k\wedge\omega^k{}_j$ where $\wedge$ denotes usual exterior matrix multiplication.

The connection one form ω as well as the curvature two-form Ω live on the given principal *G*-bundle *P*. We, now, try to obtain the corresponding objects on the base manifold *M*. For this, we consider a chart (*U,x*) at a point $p\in M$ and define a section $\sigma:U\to P$. This section induces (i) the pullback of the connection ω on U which is a Lie algebra valued one-form $\sigma^*\omega$ viz. the Yang Mills field $\Gamma=\sigma^*\omega\in\Omega^1(M)\otimes T_eG$ (which is not a tensor) and (ii) the pullback of the curvature $\sigma^*\Omega$ called the Riemannian (*Riem*) or the Yang Mills field strength which is a Lie algebra valued two-form $Riem=\sigma^*\Omega\in\Omega^2(M)\otimes T_eG$. In passing, we note that $\sigma^*\Omega=\sigma^*d\omega+\sigma^*\omega\wedge\sigma^*\omega$ so that $Riem=d\Gamma+\Gamma\wedge\Gamma$. By the Bianchi's first identity, we have $D\Omega=0$.

***16.3 The Solder Form**:* Consider a principal *G*-bundle $P\xleftarrow{\triangleleft G}P\xrightarrow{\pi}M$ and let ω be a connection one-form on *P* and let *V* be a linear representation space of *G*. Then, a *V*-valued one form θ on *P* is called a solder form if (i) θ is a *V*-valued one form i.e. $\theta\in\Omega^1(P)\otimes V$ where *V*is a linear representation space of *G* with dim *V*=dim *M*. (ii) $\theta(ver(X))=0\ \forall X\in\Gamma(TP)$ (iii) $(\triangleleft g)^*\theta=\theta$. This is the *G*-equivariant condition that ensures that if one moves the solder form from a point in a fibre to another point in the same fibre it remains invariant. (iv) There exists an associated bundle isomorphism between the tangent bundle and the associated bundle $P_V\ TM\cong_{associated\ bundle\ isomorphism}P_V$ This makes sense because *G* is supposed to act on *V* anyway as *V* is the linear representation space of *G*. Thus, the solder form enables an identification of the linear representation space *V* of *G* with each tangent space of *M*.

Thus, while the connection one-form ω is Lie algebra valued and it annihilates a horizontal vector field, the solder form θ is a linear representation space valued one-form that annihilates the vertical vector fields.

We illustrate the concept of solder form by reference to the frame bundle *P*= *LM* as a principal $GL(d,R)$-bundle with base manifold *M* of dimension *d*. Let *e* be an arbitrary frame with the corresponding coframe ε . We have $\theta:\Gamma(TP)\to R^d\equiv V$ defined by $\theta_e(X)=(u_e^{-1}\circ\pi_*)(X)$ where $u_e:R^d\xrightarrow{\sim}T_{\pi(e)}M$ as $\left(0,0,..,0,\underset{i^{th}\ entry}{1},0,..,0,0\right)\mapsto e_i$ so that $u_e^{-1}:T_{\pi(e)}M\xrightarrow{\sim}R^d$ takes a vector $Z\mapsto\varepsilon(Z)$ i.e. its components in the base manifold. This is easily seen. Given a tangent vector *X* in the principal bundle *TP*, π_*X would be tangent vector in the base space whence $(u_e^{-1}\circ\pi_*)(X)$ are the components of the tangent vector in the base space *M*.

***16.4 Torsion**:* Consider a principal *G*-bundle $P\xleftarrow{\triangleleft G}P\xrightarrow{\pi}M$ and let ω , θ be the connection one-form and solder form on *P*. Then, the torsion Θ is *V*-valued two-form defined by $\Theta=D\theta\in\Omega^2(P)\otimes V$. We have $\Theta=d\theta+\omega\wedge^*\theta$ where $\wedge^*$ is the action of the Lie algebra valued ω on the *V* (linear representation space) valued θ. For a matrix group $\Theta^i=d\theta^i+\omega^i{}_k\wedge^*\theta^k$. The second Bianchi identity for torsion gives $D\Theta=\Omega\wedge^*\theta$. $D\Theta$ is *V*-valued three-form i.e. $D\Theta\in\Omega^3(P)\otimes V$. As in the case of curvature, Ω , Θ resides in the principal bundle *P* but we can obtain its pullback to the base manifold and define $T=\sigma^*\Theta\in\Omega^2(M)\otimes V$

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17. COVARIANT DERIVATIVES

Consider a principal *G*-bundle $P\xleftarrow{\triangleleft G}P\xrightarrow{\pi}M$ and let ω be the connection one-form on *P*. Let $P_F\xrightarrow{\pi_F}M$ be an associated vector bundle on which we define the left action of *G*as $\triangleright:G\times F\to F$ where *F* is a vector space equipped with the usual addition and scalar multiplication. This left action is linear i.e. $G\triangleright:F\xrightarrow{\sim}F$ (*F* being a vector space) The connection ω on *P* effects a parallel transport between the fibres of *P*/ We define a section $\sigma:M\to P_F\ (\pi_F\circ\sigma=id_M)$ on the associated bundle and claim that there exists a unique *G*- equivariant function $\phi_\sigma:P\to F$ on the principal bundle such that $\sigma\mapsto\phi_\sigma$ is a bijective correspondence. The *G*-equivariance condition mandates $\phi_\sigma(p\triangleleft g)=g^{-1}\triangleright\phi_\sigma(p)$. We establish this as follows: (i)Let a *G*-equivariant $\phi:P\to F$ be given. We construct the map $\sigma_\phi:M\to P_F$ defined at every point $x\in M$ by $\sigma_\phi(x)=[p,\phi(p)]$ where $p\in\pi^{-1}(\{x\})$. This map is well defined as given $p'\in\pi^{-1}(\{x\}),p'\neq p,$ there exists a unique $g\in G|p'=p\triangleleft g$. Then $[p',\phi(p')]=[p\triangleleft g,\phi(p\triangleleft g)]$ $=[p\triangleleft g,g^{-1}\triangleright\phi(p)]\ =_{def\ of\ equivalence\ class}[p,\phi(p)]$. Also, $\pi_F\circ\sigma_\phi(x\in M)=\pi_F([p,\phi(p)])=\pi(p)=x$ because $p\in\pi^{-1}(\{x\})$ confirming that σ_ϕ is, indeed a section. (ii) Let the section $\sigma:M\to P_F$ on the associated bundle be given. We construct the map $\phi_\sigma:P\to F$ by $\phi_\sigma(p)=i_p^{-1}(\sigma(\pi(p)))$ where the map $i_p:F\to\pi_F^{-1}(\pi(p))\subset P_F$ is defined by $i_p(f)=[(p,f)]$. i_p is a bijection. We, also, have $i_p(f)=[(p,f)]=[(p\triangleleft g,g^{-1}\triangleright f)]=i_{p\triangleleft g}(g^{-1}\triangleright f)$. To show the *G*-equivariance of ϕ_σ , we have $\phi_\sigma(p\triangleleft g)$ $=i^{-1}_{(p\triangleleft g)}(\sigma(\pi(p\triangleleft g)))=i^{-1}_{(p\triangleleft g)}(\sigma(\pi(p)))=i^{-1}_{(p\triangleleft g)}(i_p(\phi_\sigma(p)))$ $=i^{-1}_{(p\triangleleft g)}(i_{p\triangleleft g}(g^{-1}\triangleright\phi_\sigma(p)))=g^{-1}\triangleright\phi_\sigma(p)$ thereby establishing *G*-equivariance of $\phi_\sigma:P\to F$ (iii) We, finally, establish that $\sigma_{\phi_{\sigma'}}=\sigma'$ for any given sections σ , σ' on the associated bundle and similarly, $\phi_{\sigma_{\phi'}}=\phi'$ for functions ϕ , ϕ' on the principal bundle. We have $\sigma_{\phi_{\sigma'}}(x\in M)$ $=[p,\phi_{\sigma'}]$ for some $p\in\pi^{-1}(\{x\})$ $=[p,i_p^{-1}(\sigma'(\pi(p)))]$ $=i_p(i_p^{-1}(\sigma'(\pi(p))))=\sigma'(\pi(p))=\sigma'(x)$. Similarly $\phi_{\sigma_{\phi'}}(p\in P)$

$= i_p^{-1}\left(\sigma_{\phi'}\left(\pi(p)\right)\right) = i_p^{-1}\left(\left[p, \phi'(p)\right]\right) = i_p^{-1}\left(i_p\left(\phi'(p)\right)\right) = \phi'(p)$ as required.

We, now, make use of the *G*-equivariance to establish an important result. We have, $\phi(p \triangleleft g) = g^{-1} \triangleright \phi(p)$. Expressing $g \in G$ in terms of its Lie algebra generator, say $A \in T_eG$, we obtain $\phi(p \triangleleft \exp(At)) = \exp(-At) \triangleright \phi(p)$. In the context of the linear left action $G \triangleright : F \xrightarrow{\sim} F$ where F is a finite dimensional vector space, we have $\frac{d}{dt}\phi(p \triangleleft \exp(At))\Big|_{t=0} = d\phi(p)(X_p^A)$, $\frac{d}{dt}\left(\exp(-At) \triangleright \phi(p)\right)\Big|_{t=0}$ $= -A \triangleright \phi(p) = -\omega(X_p^A) \triangleright \phi(p)$ or $d\phi(X^A) + \omega(X^A) \triangleright \phi$ $= 0$ for a linear left action.

We define the covariant derivative $\nabla_T \sigma \in \Gamma(P_F)$ in the associated bundle with respect to a section $\sigma : M \to P_F$ of a tangent vector T at a point $x \in M$ i.e. $T \in T_xM$ satisfying (i) $\nabla_{fT+S}\sigma = f\nabla_T\sigma + \nabla_S\sigma \; \forall f \in C^\infty(M), S, T \in T_xM$ (ii) $\nabla_T(\sigma + \tau)$ $= \nabla_T\sigma + \nabla_T\tau \; \forall \sigma, \tau \in \Gamma(P_F)$ (iii) $\nabla_T(f\sigma) = (Tf)\sigma +$ $f\nabla_T\sigma \;\; \forall f \in C^\infty(M), \sigma \in \Gamma(P_F)$.

We, now, need to construct an equivalent structure on the base manifold *M*. For this purpose, we proceed in two steps: (i) Let a section $\sigma : M \to P_F$ on the associated bundle be given. We can, then, construct the map $\phi_\sigma : P \to F$ by $\phi_\sigma(p) = i_p^{-1}\left(\sigma\left(\pi(p)\right)\right)$ that lies on the principal bundle as explained above. This map is bijective. $\phi_\sigma \equiv \phi : P \to F$ is a *G*-invariant fibre valued zero-form on the principal bundle *P*. We define the covariant derivative $D\phi$ by its action on a vector or pointwise action on a vector field as $D\phi(X) = d\phi \circ \left(hor(X)\right)$, $X \in T_pP, p \in P$. We establish that $D\phi(X) = d\phi(X) + \omega(X) \triangleright \phi(X)$. First, we assume X to be purely vertical. Then there exists a unique Lie algebra generator A of X i.e. there exists unique $A \in T_eG \,|\, X = X^A$. Now, LHS is $D\phi(X) = d\phi\left(hor(X)\right) = 0$ because X being vertical, its horizontal component vanishes. $d\phi(X) + \omega(X) \triangleright \phi(X) = 0$ by the property of *G*-equivariance proved above. Now, let X be horizontal, so that LHS is $D\phi(X) = d\phi\left(hor(X)\right) = d\phi(X)$ whereas the RHS is $d\phi(X) + \omega(X) \triangleright \phi(X) = d\phi(X)$ because $\omega(X)$ vanishes for a horizontal field. We write $D\phi(X) = D_X(\phi)$ for $X \in T_pP$ and note that this is linear in *X*. (ii) So far we have computed a covariant derivative that acts on vectors in the principal bundle. However, we want a covariant derivative that acts on tangent vectors in the base manifold. To obtain this structure, we follow the usual scheme of taking pullbacks of the various quantities involved. Hence, we start by defining a local section $\xi : U \to P$ where U is an open subset of *M*. We define the pullback of $\phi : P \to F$ by the section $\xi : U \to P$ as $s \equiv \xi^*\phi : TM \to F$ which is a local *F*-valued function. Similarly, the connection $\omega \in \Omega^1(P) \otimes T_eG$ has the pullback under ξ as the Yang Mills field $\xi^*\omega \in \Omega^1(M) \otimes T_eG$ and $D\phi \in \Omega^1(P) \otimes F$ has the pullback $\xi^*(D\phi) \in \Omega^1(M) \otimes F$. Putting these pieces together, $\nabla_T s(T) = \xi^*(D\phi)(T) = \xi^*(d\phi + \omega \triangleright \phi)(T) \quad T \in T_xM$ $= \xi^*(d\phi)(T) + \xi^*(\omega \triangleright \phi)(T) = d(\xi^*\phi)(T) + (\xi^*\omega)(T) \triangleright (\xi^*\phi)(T)$ $= ds(T) + \omega^{U,\xi}(T) \triangleright s(T)$.

18. Conclusion

An attempt has been made in this article to present a pedagogical introduction to the theory of fibre bundles from the perspective of a physicist assuming a minimal knowledge of topology and algebra. Emphasis has been laid on the conceptual development of the subject in a manner that is amenable to physical applications. Some contemporary applications of fibre bundles in physics and econophysics are proposed to be covered in a sequel to this write-up. As is to be expected in a work of this nature, no originality over the contents is claimed, although some innovative features are definitely embedded in the pedagogy and the presentation.

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