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A Study on the Hyperbola $x^2 - 8xy + y^2 - 6x - 6y + 18 = 0$

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ABSTRACT

The binary quadratic equation $x^2 - 8xy + y^2 - 6x - 6y + 18 = 0$ representing hyperbola is studied for its non-trivial solutions. The recurrence relations satisfied by the solutions x and y are given. A few interesting properties among the solutions are presented.

Keywords: Binary quadratic equation and Integral solutions.

1. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-20], the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. However, in [14] it is shown that the hyperbola represented by $3x^2 + xy = 14$ has only finite number of integral points. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by $x^2 - 8xy + y^2 - 6x - 6y + 18 = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$x^2 - 8xy + y^2 - 6x - 6y + 18 = 0 \qquad (1)$$

It is to be noted that (1) represents a hyperbola. By shifting the origin to the center (1,1), (1) reduces to

$$X^2 - 8XY + Y^2 = -24 \qquad (2)$$

where $x = X - 1, y = Y - 1$ (3)

Again setting

$$X = M + 3N, Y = M - 3N \qquad (4)$$

In (2), it simplifies to the equation

$$M^2 = 15N^2 + 4 \qquad (5)$$

Now, consider the Pellian equation

$$M^2 = 15N^2 + 1 \qquad (6)$$

Whose general solution $(\tilde{N}_n, \tilde{M}_n)$ is given by

$$\tilde{N}_n = \frac{1}{2\sqrt{15}}\left[\left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1}\right],$$

$$\tilde{M}_n = \frac{1}{2}\left[\left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}\right], \quad n = 0,1,2,...$$

Thus, the general solutions $(\tilde{N}_n, \tilde{M}_n)$ of (5) is given by

$$N_n = 2\tilde{N}_n = \frac{1}{\sqrt{15}}\left[\left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1}\right]$$

$$M_n = 2\tilde{M}_n = \left[\left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}\right]$$

Taking the advantage of (3) and (4), the sequence of integral solutions of (1) can be written as

$$x_n = M_n + 3N_n - 1 = 2\tilde{M}_n + 6\tilde{N}_n - 1 \qquad (7)$$

$$y_n = M_n - 3N_n - 1 = 2\tilde{M}_n - 6\tilde{N}_n - 1, \quad n = 0,1,2,... \qquad (8)$$

Thus (7) and (8) represent the non-zero distinct integral solutions of (1). The above values of x_n and y_n satisfy respectively the following recurrence relations.

$$x_{n+2} - 8x_{n+1} + x_n = 6,$$

$$y_{n+2} - 8x_{n+1} + x_n = 6, \quad n = 0,1,2,...$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below.

Table 1: Examples

n	x_n	y_n
0	13	1
1	109	13
2	865	109
3	6817	865
4	53677	6817

From the above table, we observe some interesting relations among the solutions which are presented below.

I. Both the values of x_n , y_n are positive and odd.

II. Each of the following expressions is a Nasty Number:

1. $3(9x_{2n+1} - x_{2n+2} + 12)$
2. $\frac{3}{8}(71x_{2n+1} - x_{2n+3} + 102)$
3. $3(x_{2n+1} + y_{2n+1} + 6)$
4. $3(9x_{2n+1} - y_{2n+3} + 12)$
5. $3(71x_{2n+2} - 9x_{2n+3} + 66)$
6. $\frac{3}{8}(x_{2n+2} + 9x_{2n+1} + 42)$
7. $3(9y_{2n+2} - x_{2n+2} + 12)$

8. $\frac{1}{21}(x_{2n+3}+71y_{2n+1}+324)$
9. $\frac{3}{8}(71y_{2n+2}-x_{2n+3}+102)$
10. $3(71y_{2n+3}-9x_{2n+3}+66)$
11. $3(y_{2n+1}+y_{2n+2}+6)$
12. $\frac{3}{8}(9y_{2n+1}+y_{2n+3}+42)$
13. $3(9y_{2n+2}-y_{2n+3}+12)$

III. Each of the following expressions is a Cubical Integer:

1. $4(9x_{3n+2}-x_{3n+3}+27x_n-3x_{n+1}+32)$
2. $256(71x_{3n+2}-x_{3n+4}+213x_n-3x_{n+2}+280)$
3. $4(x_{3n+2}+y_{3n+2}+3x_n+3y_n+8)$
4. $4(9x_{3n+2}-y_{3n+4}+27x_n-3y_{n+2}+32)$
5. $4(71x_{3n+3}-9x_{3n+4}+213x_{n+1}-27x_{n+2}+248)$
6. $256(x_{3n+3}+9x_{3n+2}+3x_{n+1}+27x_n+40)$
7. $4(9y_{3n+3}-x_{3n+3}+27y_{n+1}-3x_{n+1}+32)$
8. $15876(x_{3n+4}+71y_{3n+2}+3x_{n+2}+213y_n+288)$
9. $256(71y_{3n+3}-x_{3n+4}+213y_{n+1}-3x_{n+2}+280)$
10. $4(71y_{3n+4}-9x_{3n+4}+213y_{n+2}-27x_{n+2}+248)$
11. $4(y_{3n+2}+y_{3n+3}+3y_n+3y_{n+1}+8)$
12. $256(9y_{3n+2}+y_{3n+4}+27y_n+3y_{n+2}+40)$
13. $4(9y_{3n+3}-y_{3n+4}+27y_{n+1}-3y_{n+2}+32)$

IV. Each of the following expressions is a Bi-Quadratic Integer:

1. $8(9x_{4n+3}-x_{4n+4}+36x_{2n+1}-4x_{2n+2}+52)$
2. $4096(71x_{4n+3}-x_{4n+5}+284x_{2n+1}-4x_{2n+3}+446)$
3. $8(x_{4n+3}+y_{4n+3}+4x_{2n+1}+4y_{2n+1}+22)$
4. $8(9x_{4n+3}-y_{4n+5}+36x_{2n+1}-4y_{2n+3}+52)$
5. $8(71x_{4n+4}-9x_{4n+5}+284x_{2n+2}-36x_{2n+3}+322)$
6. $4096(x_{4n+4}+9y_{4n+3}+4x_{2n+2}+36y_{2n+1}+146)$
7. $8(9y_{4n+4}-x_{4n+4}+36y_{2n+2}-4x_{2n+2}+52)$
8. $2000376(x_{4n+5}+71y_{4n+3}+4x_{2n+3}+284y_{2n+1}+1116)$
9. $4096(71y_{4n+4}-x_{4n+5}+284y_{2n+2}-4x_{2n+3}+446)$
10. $8(71y_{4n+5}-9x_{4n+5}+284y_{2n+3}-36x_{2n+3}+322)$
11. $8(y_{4n+3}+y_{4n+4}+4y_{2n+1}+4y_{2n+2}+22)$
12. $4096(9y_{4n+3}+y_{4n+5}+36y_{2n+1}+4y_{2n+3}+146)$
13. $8(9y_{4n+4}-y_{4n+5}+36y_{2n+2}-4y_{2n+3}+52)$

V. Relations among the solutions:

1. $x_n=y_{n+1}$
2. $x_n=8x_{n+1}-x_{n+2}+6$
3. $x_{n+1}=8x_n-y_n+6$
4. $x_{n+1}=y_{n+2}$
5. $63x_{n+1}=8x_{n+2}+y_n-54$
6. $x_{n+1}=8y_{n+1}-y_n+6$
7. $x_{n+2}=63x_n-8y_n+54$
8. $x_{n+2}=63y_{n+1}-8y_n+54$
9. $x_{n+2}=63y_{n+2}-y_n+54$
10. $y_n=8x_n-y_{n+2}+6$
11. $y_{n+1}=8x_{n+1}-x_{n+2}+6$
12. $8y_{n+2}=x_n+x_{n+2}-6$
13. $8y_{n+2}=y_{n+1}+x_{n+2}-6$
14. $y_{n+2}=8y_{n+1}-y_n+6$

3. REMARKABLE OBSERVATIONS

I) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table 2 below.

Table 2: Hyperbola

S. No.	(X, Y)	Hyperbola
1.	$(9x_n-x_{n+1}+8,\ x_{n+1}-7x_n-6)$	$3X_n^2-5Y_n^2=48$
2.	$(71x_n-x_{n+2}+70,\ x_{n+2}-55x_n-54)$	$3X_n^2-5Y_n^2=3072$
3.	$(x_n+y_n+2,\ x_n-y_n)$	$3X_n^2-5Y_n^2=48$
4.	$(9x_n-y_{n+2}+8,\ y_{n+2}-7x_n-6)$	$3X_n^2-5Y_n^2=48$
5.	$(213x_{n+1}-27x_{n+2}+186,\ 7x_{n+2}-55x_{n+1}-48)$	$3X_n^2-5Y_n^2=48$
6.	$(x_{n+1}+9y_n+10,\ x_{n+1}-7y_n-6)$	$3X_n^2-5Y_n^2=3072$
7.	$(9y_{n+1}-x_{n+1}+8,\ x_{n+1}-7y_{n+1}-6)$	$3X_n^2-5Y_n^2=48$
8.	$(x_{n+2}+71y_n+72,\ x_{n+2}-55y_n-54)$	$3X_n^2-5Y_n^2=190512$
9.	$(71y_{n+1}-x_{n+2}+70,\ x_{n+2}-55y_{n+1}-54)$	$3X_n^2-5Y_n^2=3072$
10.	$(71y_{n+2}-9x_{n+2}+62,\ 7x_{n+2}-55y_{n+2}-48)$	$3X_n^2-5Y_n^2=48$
11.	$(y_n+y_{n+1}+2,\ y_{n+1}-y_n)$	$3X_n^2-5Y_n^2=48$
12.	$(9y_n+y_{n+2}+10,\ y_{n+2}-7y_n-6)$	$3X_n^2-5Y_n^2=3072$
13.	$(9y_{n+1}-y_{n+2}+8,\ y_{n+2}-7y_{n+1}-6)$	$3X_n^2-5Y_n^2=48$

II) Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below.

Table 3: Parabolas

S. No.	(X, Y)	Parabola
1.	$\begin{pmatrix} 9x_{2n+1} - x_{2n+2} + 12, \\ x_{n+1} - 7x_n - 6 \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
2.	$\begin{pmatrix} 71x_{2n+1} - x_{2n+3} + 102, \\ x_{n+2} - 55x_n - 54 \end{pmatrix}$	$48X_n - 5Y_n^2 = 3072$
3.	$\begin{pmatrix} x_{2n+1} + y_{2n+1} + 6, \\ x_n - y_n \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
4.	$\begin{pmatrix} 9x_{2n+1} - y_{2n+3} + 12, \\ y_{n+2} - 7x_n - 6 \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
5.	$\begin{pmatrix} 71x_{2n+2} - 9y_{2n+3} + 66, \\ 7x_{n+2} - 55x_{n+1} - 48 \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
6.	$\begin{pmatrix} x_{2n+2} + 9y_{2n+1} + 42, \\ x_{n+1} - 7y_n - 6 \end{pmatrix}$	$X_n - 5Y_n^2 = 64$
7.	$\begin{pmatrix} 9y_{2n+2} - x_{2n+2} + 12, \\ x_{n+1} - 7y_{n+1} - 6 \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
8.	$\begin{pmatrix} x_{2n+3} + 71y_{2n+1} + 324, \\ x_{n+2} - 55y_n - 54 \end{pmatrix}$	$378X_n - 5Y_n^2 = 190512$
9.	$\begin{pmatrix} 71y_{2n+2} - x_{2n+3} + 102, \\ x_{n+2} - 55y_{n+1} - 54 \end{pmatrix}$	$48X_n - 5Y_n^2 = 3072$
10.	$\begin{pmatrix} 71y_{2n+3} - 9x_{2n+3} + 66, \\ 7x_{n+2} - 55y_{n+2} - 48 \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
11.	$\begin{pmatrix} y_{2n+1} + y_{2n+2} + 6, \\ y_{n+1} - y_n \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$
12.	$\begin{pmatrix} 9y_{2n+1} + y_{2n+3} + 42, \\ y_{n+2} - 7y_n - 6 \end{pmatrix}$	$48X_n - 5Y_n^2 = 3072$
13.	$\begin{pmatrix} 9y_{2n+2} - y_{2n+3} + 12, \\ y_{n+2} - 7y_{n+1} - 6 \end{pmatrix}$	$6X_n - 5Y_n^2 = 48$

4. CONCLUSION

In conclusion, one may search for other patterns of solutions and their corresponding properties.

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