Observations on the Hyperbola $y^2 = 117x^2 + 1$

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ABSTRACT

The binary quadratic equation represented by the positive pellian $y^2 = 117x^2 + 1$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary quadratic, Hyperbola, Integral solutions, Parabola and Pell equation.

1. INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 117x^2 + 1$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Table 1: Numerical Examples

n	x_n	y_n
0	60	649
1	77880	842401
2	101088180	1093435849
3	131212379760	1419278889601
4	170313567840300	1842222905266249

2. METHOD OF ANALYSIS

The positive pell equation representing hyperbola under consideration is,

$$y^2 = 117x^2 + 1 \qquad (1)$$

The smallest positive integer solutions of (1) is

$x_0 = 60, y_0 = 649$

The general solution (x_n, y_n) of (1) is given by

$$y_n = \frac{1}{2} f_n, x_n = \frac{g_n}{2\sqrt{117}} \qquad (2)$$

Where,

$$f_n = \left(649 + 60\sqrt{117}\right)^{n+1} + \left(649 - 60\sqrt{117}\right)^{n+1},$$
$$g_n = \left(649 + 60\sqrt{117}\right)^{n+1} - \left(649 - 60\sqrt{117}\right)^{n+1}, n = 0,1,2,3,.....$$

The recurrence relations satisfied by the solutions (2) are given by

$$y_{n+2} - 1298y_{n+1} + y_n = 0$$
$$x_{n+2} - 1298x_{n+1} + x_n = 0$$

From the above, we observe some interesting relations among the solutions which are presented below:

1. x_n values are always even.
2. y_n values are always odd.
3. Each of the following expressions is a Nasty Number:
   - $12(y_{2n+1} + 1)$
   - $12(1298y_{2n+2} - y_{2n+3} + 1)$
   - $\frac{12(y_{2n+2} 7020x_{2n+1} + 649)}{649}$
   - $12(649y_{2n+2} - 7020x_{2n+2} + 1)$
   - $\frac{6(1684802y_{2n+2} - 14040x_{2n+3} + 1298)}{649}$
   - $\frac{12(y_{2n+3} - 9111960x_{2n+1} + 842401)}{842401}$
   - $12(y_{2n+3} - 14040x_{2n+2} + 1)$
   - $12(842401y_{2n+3} - 9111960x_{2n+3} + 1)$
   - $\frac{(x_{2n+2} - 649x_{2n+1} + 60)}{5}$
   - $\frac{(x_{2n+3} - 842401x_{2n+1} + 77880)}{6490}$
   - $\frac{(649x_{2n+3} - 842401x_{2n+2} + 60)}{5}$

4. Each of the following expressions is a Cubical Integer:
   - $2(y_{3n+2} + 3y_n)$
   - $2[1298y_{3n+3} - y_{3n+4} + 3(1298y_{n+1} - y_{n+2})]$
   - $842402[y_{3n+3} - 7020x_{3n+2} + 3(y_{n+1} - 7020x_n)]$
   - $2[649y_{3n+3} - 7020x_{3n+3} + 3(649y_{n+1} - 7020x_{n+1})]$

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- $421201\left[\begin{array}{l}1684802y_{3n+3}-14040x_{3n+4}\\+3(1684802y_{n+1}-14040x_{n+2})\end{array}\right]$
- $1419278889602\left[\begin{array}{l}y_{3n+4}-9111960x_{3n+2}\\+3(y_{n+2}-9111960x_{n})\end{array}\right]$
- $2[y_{3n+4}-14040x_{3n+3}+3(y_{n+2}-14040x_{n+1})]$
- $2\left[\begin{array}{l}842401y_{3n+4}-9111960x_{3n+4}\\+3(842401y_{n+2}-9111960x_{n+2})\end{array}\right]$
- $900[x_{3n+3}-649x_{3n+2}+3(x_{n+1}-649x_{n})]$
- $1516323600\left[\begin{array}{l}x_{3n+4}-842401x_{3n+2}\\+3(x_{n+2}-842401x_{n})\end{array}\right]$
- $900\left[\begin{array}{l}649x_{3n+4}-842401x_{3n+3}\\+3(649x_{n+2}-842401x_{n+1})\end{array}\right]$

5. Relations among the solutions:

- $y_{n+2}=1298y_{n+1}-y_{n}$
- $7020x_{n}=y_{n+1}-649y_{n}$
- $7020x_{n+1}=649y_{n+1}-y_{n}$
- $7020x_{n+2}=842401y_{n+1}-649y_{n}$
- $9111960x_{n}=y_{n+2}-842401y_{n}$
- $14040x_{n+1}=y_{n+2}-y_{n}$
- $9111960x_{n+2}=842401y_{n+2}-y_{n}$
- $649y_{n+1}=7020x_{n+1}+y_{n}$
- $649x_{n}=x_{n+1}-60y_{n}$
- $649x_{n+2}=60y_{n}+842401x_{n+1}$
- $7020x_{n}=649y_{n+2}-842401y_{n+1}$
- $7020x_{n+1}=y_{n+2}-649y_{n+1}$
- $7020x_{n+2}=649y_{n+2}-y_{n+1}$
- $649x_{n+1}=60y_{n+1}+x_{n}$
- $x_{n+2}=120y_{n+1}+x_{n}$
- $x_{n+2}=60y_{n+1}+649x_{n+1}$
- $842401x_{n+1}=60y_{n+2}+649x_{n}$
- $842401x_{n+2}=77880y_{n+2}-x_{n}$
- $649x_{n+2}=60y_{n+2}+x_{n+1}$
- $x_{n}=1298x_{n+1}-x_{n+2}$
- $77880y_{n}=x_{n+2}-842401x_{n}$

3. REMARKABLE OBSERVATION

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table 2 below:

Table 2: Hyperbolas

S. No.	(X,Y)	Hyperbola
1	$(y_{n+1}-649y_{n},y_{n})$	$421200Y^2-X^2=421200$
2	$(y_{n+2}-842401y_{n},y_{n})$	$709639444800Y^2-X^2=709639444800$
3	$(x_{n+1}-60y_{n},649y_{n})$	$Y^2-117X^2=421201$
4	$(x_{n+2}-77880y_{n},y_{n})$	$709639444801Y^2-117X^2=709639444801$
5	$(649y_{n+2}-842401y_{n+1},\ 1298y_{n+1}-y_{n+2})$	$421200Y^2-X^2=421200$
6	$(x_{n},y_{n+1}-7020x_{n})$	$Y^2-49280517X^2=421201$
7	$(649x_{n+1}-60y_{n+1},\ 649y_{n+1}-7020x_{n+1})$	$Y^2-117X^2=1$
8	$(x_{n+2}-120y_{n+1},\ 1684802y_{n+1}-14040x_{n+2})$	$Y^2-197122068X^2=1684804$
9	$(x_{n},y_{n+2}-9111960x_{n})$	$Y^2-83027815041717X^2=709639444801$
10	$(842401x_{n+1}-60y_{n+2},\ y_{n+2}-14040x_{n+1})$	$421201Y^2-117X^2=421201$
11	$(842401x_{n+2}-77880y_{n+2},\ 842401y_{n+2}-9111960x_{n+2})$	$Y^2-117X^2=1$
12	$(x_{n},x_{n+1}-649x_{n})$	$Y^2-421200X^2=3600$
13	$(x_{n},x_{n+2}-842401x_{n})$	$Y^2-709639444800X^2=6065294400$
14	$(1298x_{n+1}-x_{n+2},\ 649x_{n+2}-842401x_{n+1})$	$Y^2-421200X^2=3600$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table 3 below:

Table 3: Parabolas

S. No.	(X,Y)	Hyperbola
1	$(y_{n+1}-649y_{n},y_{2n+1}+1)$	$X^2=210600Y-421200$
2	$(y_{n+2}-842401y_{n},y_{2n+1}+1)$	$X^2=\left(\begin{array}{l}354819722400Y\\-709639444800\end{array}\right)$
3	$(x_{n+1}-60y_{n},y_{2n+1}+1)$	$234X^2=421201Y-842402$
4	$(x_{n+2}-77880y_{n},y_{2n+1}+1)$	$234X^2=\left(\begin{array}{l}709639444801Y\\-1419278889602\end{array}\right)$
5	$(649y_{n+2}-842401y_{n+1},\ 1298y_{2n+2}-y_{2n+3}+1)$	$X^2=210600Y-421200$
6	$(x_{n},y_{2n+2}-7020x_{2n+1}+649)$	$151866X^2=Y-1298$
7	$(649x_{n+1}-60y_{n+1},\ 649y_{2n+2}-7020x_{2n+2}+1)$	$234X^2=Y-2$

8	$\left(x_n, \left(\begin{matrix}1684802y_{2n+2}\\ -14040x_{2n+3}+1298\end{matrix}\right)\right)$	$303732X^2 = Y - 2596$
9	$\left(x_n, \left(\begin{matrix}y_{2n+3} - 9111960x_{2n+1}\\ +842401\end{matrix}\right)\right)$	$197121834X^2 = (Y - 1684802)$
10	$\left(\begin{matrix}842401x_{n+1} - 60y_{n+2},\\ y_{2n+3} - 14040x_{2n+2} + 1\end{matrix}\right)$	$234X^2 = 421201Y - 842402$
11	$\left(\begin{matrix}842401x_{n+2} - 77880y_{n+2},\\ 842401y_{2n+3} - 9111960x_{2n+3}\\ +1\end{matrix}\right)$	$234X^2 = Y - 2$
12	$(x_n, x_{2n+2} - 649x_{2n+1} + 60)$	$14040X^2 = Y - 120$
13	$\left(\begin{matrix}x_n, x_{2n+3}\\ -842401x_{2n+1} + 77880\end{matrix}\right)$	$18223920X^2 = Y - 155760$
14	$\left(\begin{matrix}1298x_{n+1} - x_{n+2},\\ 649x_{2n+3} - 842401x_{2n+2} + 60\end{matrix}\right)$	$14040X^2 = Y - 120$

III. Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}$ .Observe that $p > q > 0$ .Treat p, q as the generators of the Pythagorean Triangle $T(\alpha, \beta, \gamma)$,where $\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2$

Then the following interesting relations are observed:

a) $2\alpha - 117\beta + 115\gamma = -2$

b) $119\alpha - 2\gamma + 2 = 468\frac{A}{P}$

c) $\frac{2A}{P} = x_{n+1}y_{n+1}$

4. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 117x^2 + 1$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive pell equations and determine their integer solutions along with suitable properties.

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