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On Non-Homogeneous Sextic Equation with Five Unknowns $2(x+y)(x^3-y^3)=39(z^2-w^2)T^4$

S.Vidhyalakshmi[1], M.A.Gopalan[2] and S.Aarthy Thangam[3]

[1]*Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu, India.*
[2]*Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu, India.*
[3]*Research Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu, India.*

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ABSTRACT

The non-homogeneous sextic equation with five unknowns given by $2(x+y)(x^3-y^3)=39(z^2-w^2)T^4$ *is considered and analysed for its non-zero distinct integer solutions. Employing the linear transformations* $x=u+v, y=u-v, z=2u+v, w=2u-v, (u \neq v \neq 0)$ *and applying the method of factorization, three different patterns of non-zero distinct integer solutions are obtained. A few interesting relations between the solutions and special numbers namely Four dimensional numbers, Polygonal numbers, Octahedral numbers, Centered Pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, Kynea numbers and Star numbers are presented.*

Keywords: *Integer solutions, Non-homogeneous sextic equation and Sextic equation with five unknowns.*

1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4] particularly, in [5, 6] Sextic equations with three unknowns are studied for their integral solutions. [7-12] analyze Sextic equations with four unknowns for their non-zero integer solutions. [13-16] analyze Sextic equations with five unknowns for their non-zero integer solutions. This communication analyzes a Sextic equation with five unknowns given by $2(x+y)(x^3-y^3)=39(z^2-w^2)T^4$. Infinitely many Quintuples (x, y, z, w, T) satisfying the above equation is obtained. Various interesting properties among the values of x, y, z, w and T are presented.

2. NOTATIONS

- Polygonal number of rank n with size m

$$T_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

- Star number of rank n

$$S_n=6n(n-1)+1$$

- Octahedral number of rank n

$$OH_n=\frac{1}{3}\left(n(2n^2+1)\right)$$

- Centered Pyramidal number of rank n with size m

$$CP_{m,n}=\frac{m(n-1)n(n+1)+6n}{6}$$

- Four dimensional Figurate number of rank n whose generating polygon is a pentagon

$$F_{4,n,5}=\frac{3n^4+10n^3+9n^2+2n}{4!}$$

- Jacobsthal number of rank n

$$J_n=\frac{1}{3}\left(2^n-(-1)^n\right)$$

- Jacobsthal-Lucas number of rank n

$$j_n=2^n+(-1)^n$$

- Kynea number of rank n

$$Ky_n=(2^n+1)^2-2$$

3. METHOD OF ANALYSIS

The non-homogeneous sextic equation with five unknowns to be solved is given by

$$2(x+y)(x^3-y^3)=39(z^2-w^2)T^4 \quad (1)$$

The substitution of the linear transformations

$$x=u+v,\ y=u-v,\ z=2u+v,\ w=2u-v,\ u\neq v\neq 0 \quad (2)$$

in (1) leads to

$$3u^2+v^2=39T^4 \quad (3)$$

(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

3.1 Pattern: 1

Assume $T=T(a,b)=a^2+3b^2;\quad a,b>0$ (4)

Write 39 as

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$$39=\left(6+i\sqrt{3}\right)\left(6-i\sqrt{3}\right) \quad (5)$$

Using (4) and (5) in (3) and employing the method of factorization and equating positive factors, we get

$$\left(v+i\sqrt{3}u\right)=\left(6+i\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^4$$

Equating real and imaginary parts, we have

$$u=u(a,b)=a^4+24a^3b-18a^2b^2-72ab^3+9b^4$$

$$v=v(a,b)=6a^4-12a^3b-108a^2b^2+36ab^3+54b^4$$

Employing (2), the values of x, y, z, w and T are given by

$$x=x(a,b)=u+v=7a^4+12a^3b-126a^2b^2-36ab^3+63b^4$$

$$y=y(a,b)=u-v=-5a^4+36a^3b+90a^2b^2-108ab^3-45b^4$$

$$z=z(a,b)=2u+v=8a^4+36a^3b-144a^2b^2-108ab^3+72b^4$$

$$w=w(a,b)=2u-v=-4a^4+60a^3b+72a^2b^2-180ab^3-36b^4$$

$$T=T(a,b)=a^2+3b^2$$

Which represent non-zero distinct integer solutions of (1) in two parameters.

Properties

- $234(OH_a)-z(a,1)-2w(a,1)\equiv 0(\text{mod}2)$
- $504F_{4,b,5}-x(1,b)-246CP_{6,b}-105T_{8,b}\equiv -1(\text{mod}2)$
- $2\{T(1,b)-1\}$ is a nasty number

3.2 Pattern: 2

One may write (3) as

$$v^2+3u^2=39T^4*1 \quad (6)$$

Also, write 1 as

$$1=\frac{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}{4} \quad (7)$$

Substituting (4), (5) and (7) in (6) and employing the method of factorization and equating positive factors we get

$$\left(v+i\sqrt{3}u\right)=\frac{\left(6+i\sqrt{3}\right)\left(1+i\sqrt{3}\right)}{2}\left(a+i\sqrt{3}b\right)^4$$

Equating real and imaginary parts, we have

$$u=u(a,b)=\frac{1}{2}\left(7a^4+12a^3b-126a^2b^2-36ab^3+63b^4\right) \quad (8)$$

$$v=v(a,b)=\frac{1}{2}\left(3a^4-84a^3b-54a^2b^2+252ab^3+27b^4\right) \quad (9)$$

The choices a=2A and b=2B in (8), (9) lead to

$$u=u(A,B)=56A^4+96A^3B-1008A^2B^2-288AB^3+504B^4$$

$$v=v(A,B)=24A^4-672A^3B-432A^2B^2+2016AB^3+216B^4$$

In view of (2), the integer values of x, y, z, w and T are given by

$$x=x(A,B)=80A^4-576A^3B-1440A^2B^2+1728AB^3+720B^4$$

$$y=y(A,B)=32A^4+768A^3B-576A^2B^2-2304AB^3+288B^4$$

$$z=z(A,B)=136A^4-480A^3B-2448A^2B^2+1440AB^3+1224B^4$$

$$w=w(A,B)=88A^4+864A^3B-1584A^2B^2-2592AB^3+792B^4$$

$$T=T(A,B)=4A^2+12B^2$$

Which represent non-zero distinct integer solutions of (1) in two parameters.

Properties

- $w(1,B)-x(1,B)-72(T_{4,B})^2+4320CP_{6,B}+24S_B\equiv 0(\text{mod}2)$
- $T(2^n,1)+12J_n+4j_n-16=4Ky_n$
- $3\{x(A,A)-y(A,A)-2z(A,A)\}$ is a nasty number

Remark

It is worth to note that 39 in (5) and 1in (7) are also represented in the following ways

$$39=\frac{\left(3+i7\sqrt{3}\right)\left(3-i7\sqrt{3}\right)}{4}=\frac{\left(9+i5\sqrt{3}\right)\left(9-i5\sqrt{3}\right)}{4} \qquad 1=\frac{\left(1+i4\sqrt{3}\right)\left(1-i4\sqrt{3}\right)}{49}=\frac{\left(11+i4\sqrt{3}\right)\left(11-i4\sqrt{3}\right)}{169}$$

By introducing the above representations in (5) and (7), one may obtain different patterns of solutions to (1).

3.3 Pattern: 3

Write (3) as

$$3\left(u^2-T^4\right)=36T^4-v^2 \quad (10)$$

Factorizing (10) we have

$$3\left(u+T^2\right)\left(u-T^2\right)=\left(6T^2+v\right)\left(6T^2-v\right) \quad (11)$$

This equation is written in the form of ratio as

$$\frac{3\left(u-T^2\right)}{6T^2-v}=\frac{\left(6T^2+v\right)}{u+T^2}=\frac{a}{b},\quad b\neq 0 \quad (12)$$

Which is equivalent to the system of double equations

$$3bu+av-(6a+3b)T^2=0 \quad (13)$$

$$-au+bv+(-a+6b)T^2=0 \quad (14)$$

Applying the method of cross multiplication, we get

$$u=-a^2+3b^2+12ab \quad (15)$$

$$v=6a^2-18b^2+6ab \quad (16)$$

$$T^2=a^2+3b^2 \quad (17)$$

Now, the solution for (17) is

$$a=3p^2-q^2,\quad b=2pq,\quad T=3p^2+q^2 \quad (18)$$

Using (18) in (15) and (16), we get

$$u=u(p,q)=-9p^4+72p^3q+18p^2q^2-24pq^3-q^4$$

$$v=v(p,q)=54p^4+36p^3q-108p^2q^2-12pq^3+6q^4$$

In view of (2), the integer values of x, y, z, w, T are given by

$$x=x(p,q)=u+v=45p^4+108p^3q-90p^2q^2-36pq^3+5q^4$$

$$y=y(p,q)=u-v=-63p^4+36p^3q+126p^2q^2-12pq^3-7q^4$$

$$z=z(p,q)=2u+v=36p^4+180p^3q-72p^2q^2-60pq^3+4q^4$$

$$w=w(p,q)=2u-v=-72p^4+108p^3q+144p^2q^2-36pq^3-8q^4$$

$$T=T(p,q)=3p^2+q^2$$

Which represent non-zero distinct integer solutions of (1) in two parameters.

Properties

- $2z(1,q)+w(1,q)+156(CP_{6,q})\equiv 0(\text{mod}3)$
- $T(2^n,1)-3ky_n+6j_n=\begin{cases}-2, & \text{if } n \text{ is odd}\\ 10, & \text{if } n \text{ is even}\end{cases}$

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- $468(CP_{6,p}) - 2x(p,1) - 2y(p,1) - z(p,1) \equiv 0(\mod 2)$

4. CONCLUSION

First of all, it is worth to mention here that in (2), the values of z and w may also be represented by $z = 2uv + 1, w = 2uv - 1$ and $z = uv + 2, w = uv - 2$ and thus, will obtain other choices of solutions to (1). In conclusion, one may consider other forms of Sextic equation with five unknowns and search for their integer solutions.

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