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# Integer Solutions on the Hyperbola $y^2 = 87x^2 + 13$

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ABSTRACT

*The binary quadratic equation represented by the positive pellian $y^2 = 87x^2 + 13$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.*

*Keywords: Binary quadratic, Hyperbola, Integral solutions, Parabola, Pell equation.*

## 1. INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 87x^2 + 13$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

## 2. METHOD OF ANALYSIS

The positive pell equation representing hyperbola under consideration is,

$$y^2 = 87x^2 + 13 \qquad (1)$$

The smallest positive integer solutions of (1) is $x_0 = 1, y_0 = 10$

The general solution $(x_n, y_n)$ of (1) is given by

$$y_n = \frac{1}{2} f_n, \quad x_n = \frac{1}{2\sqrt{87}} g_n$$

Where,

$$f_n = \left(28 + 3\sqrt{87}\right)^{n+1} + \left(28 - 3\sqrt{87}\right)^{n+1}$$
$$g_n = \left(28 + 3\sqrt{87}\right)^{n+1} - \left(28 - 3\sqrt{87}\right)^{n+1}$$

Applying Brahamagupta Lemma between $(x_0, y_0)$ and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$2\sqrt{87} x_{n+1} = \sqrt{87} f_n + 10 g_n$$
$$2y_{n+1} = 10 f_n + \sqrt{87} g_n$$

The recurrence relations satisfied by the solutions $x \& y$ are given by

$$x_{n+3} - 56x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - 56y_{n+2} + y_{n+1} = 0$$

Some numerical examples of $x \& y$ satisfying (1) are given in the Table 1 below:

Table I: Examples

| n | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 1 | 10 |
| 1 | 58 | 541 |
| 2 | 3247 | 30286 |
| 3 | 181774 | 1695475 |
| 4 | 10176097 | 94916314 |

From the above table, we observe some interesting relations among the solutions which are presented below:

1. $x_n, y_n$ values are alternatively odd and even.
2. Each of the following expressions is a nasty number

- $\frac{40x_{2n+3} - 2164x_{2n+2} + 156}{13}$
- $\frac{5x_{2n+4} - 15143x_{2n+2} + 1092}{91}$
- $\frac{30y_{2n+3} - 15138x_{2n+2} + 1092}{91}$
- $\frac{120y_{2n+4} - 3389868x_{2n+2} + 244452}{20371}$
- $\frac{2164x_{2n+4} - 121144x_{2n+3} + 156}{13}$
- $\frac{1623y_{2n+2} - 261x_{2n+3} + 1092}{91}$
- $\frac{6492y_{2n+3} - 60552x_{2n+3} + 156}{13}$
- $\frac{1623y_{2n+4} - 847467x_{2n+3} + 1092}{91}$

- $\frac{363432y_{2n+2}-1044x_{2n+4}+244452}{20371}$
- $\frac{90858y_{2n+3}-15138x_{2n+4}+1092}{91}$
- $\frac{363432y_{2n+4}-3389868x_{2n+4}+156}{13}$
- $\frac{232y_{2n+2}-4y_{2n+3}+156}{13}$
- $\frac{3247y_{2n+2}-y_{2n+4}+2184}{182}$
- $\frac{12988y_{2n+3}-232y_{2n+4}+156}{13}$

3. Each of the following expressions is a cubical integer

- $1521(20x_{3n+4}-1082x_{3n+3}+60x_{n+2}-3246x_{n+1})$
- $1192464\begin{pmatrix}10x_{3n+5}-30286x_{3n+3}+30x_{n+3}\\-90858x_{n+1}\end{pmatrix}$
- $33124(10y_{3n+4}-5046x_{3n+3}+30y_{n+2}-15138x_{n+1})$
- $414977641\begin{pmatrix}20y_{3n+5}-564978x_{3n+3}+60y_{n+3}\\-1694934x_{n+1}\end{pmatrix}$
- $1521\begin{pmatrix}1082x_{3n+5}-60572x_{3n+4}+3246x_{n+3}\\-181716x_{n+2}\end{pmatrix}$
- $33124(541y_{3n+3}-87x_{3n+4}+1623y_{n+1}-261x_{n+2})$
- $169\begin{pmatrix}1082y_{3n+4}-10092x_{3n+4}+3246y_{n+2}\\-30276x_{n+2}\end{pmatrix}$
- $33124\begin{pmatrix}541y_{3n+5}-282489x_{3n+4}+1623y_{n+3}\\-847467x_{n+2}\end{pmatrix}$
- $169\begin{pmatrix}60572y_{3n+5}-564978x_{3n+5}+181716y_{n+3}\\-1694934x_{n+3}\end{pmatrix}$
- $1521(116y_{3n+3}-2y_{3n+4}+348y_{n+1}-6y_{n+2})$
- $1192464(3247y_{3n+3}-y_{3n+5}+9741y_{n+1}-3y_{n+3})$
- $1521(6494y_{3n+4}-116y_{3n+5}+19482y_{n+2}-348y_{n+3})$

4. Relations among the solutions

- $x_{n+3}=56x_{n+2}-x_{n+1}$
- $3y_{n+1}=x_{n+2}-28x_{n+1}$
- $3y_{n+2}=28x_{n+2}-x_{n+1}$
- $3y_{n+3}=1567x_{n+2}-28x_{n+1}$
- $168y_{n+1}=x_{n+3}-1567x_{n+1}$
- $6y_{n+2}=x_{n+3}-x_{n+1}$
- $168y_{n+3}=1567x_{n+3}-x_{n+1}$
- $28y_{n+1}=y_{n+2}-261x_{n+1}$
- $28y_{n+3}=1567y_{n+2}+261x_{n+1}$
- $1567y_{n+1}=y_{n+3}-14616x_{n+1}$
- $3y_{n+1}=28x_{n+3}-1567x_{n+2}$
- $3y_{n+2}=x_{n+3}-28x_{n+2}$
- $3y_{n+3}=28x_{n+3}-x_{n+2}$
- $28y_{n+2}=y_{n+1}+261x_{n+2}$
- $y_{n+3}=28y_{n+2}+261x_{n+2}$
- $y_{n+1}=y_{n+3}-522x_{n+2}$
- $1567y_{n+2}=28y_{n+1}+261x_{n+3}$
- $1567y_{n+3}=y_{n+1}+14616x_{n+3}$
- $28y_{n+3}=y_{n+2}+261x_{n+3}$
- $y_{n+3}=56y_{n+2}-y_{n+1}$

## 3. REMARKABLE OBSERVATIONS

i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table II below.

Table II: Hyperbolas

| S.NO | $(X,Y)$ | Hyperbola |
|---|---|---|
| 1 | $(58x_{n+1}-x_{n+2},\ 10x_{n+2}-541x_{n+1})$ | $Y^2-87X^2=1521$ |
| 2 | $(3247x_{n+1}-x_{n+3},\ 10x_{n+3}-30286x_{n+1})$ | $Y^2-87X^2=4769856$ |
| 3 | $(541x_{n+1}-y_{n+2},\ 10y_{n+2}-5046x_{n+1})$ | $Y^2-87X^2=132496$ |
| 4 | $(30286x_{n+1}-y_{n+3},\ 10y_{n+3}-292489x_{n+1})$ | $Y^2-87X^2=414977641$ |
| 5 | $(3247x_{n+2}-58x_{n+3},\ 541x_{n+3}-30286x_{n+2})$ | $Y^2-87X^2=1521$ |
| 6 | $(10x_{n+2}-58y_{n+1},\ 541y_{n+1}-87x_{n+2})$ | $Y^2-87X^2=132496$ |
| 7 | $(541x_{n+2}-58y_{n+2},\ 541y_{n+2}-5046x_{n+2})$ | $Y^2-87X^2=169$ |
| 8 | $(30286x_{n+2}-58y_{n+3},\ 541y_{n+3}-282489x_{n+2})$ | $Y^2-87X^2=132496$ |
| 9 | $(10x_{n+3}-3247y_{n+1},\ 30286y_{n+1}-87x_{n+3})$ | $Y^2-87X^2=414977641$ |
| 10 | $(541x_{n+3}-3247y_{n+2},\ 30286y_{n+2}-5046x_{n+3})$ | $Y^2-87X^2=132496$ |
| 11 | $(30286x_{n+3}-3247y_{n+3},\ 30286y_{n+3}-282489x_{n+3})$ | $Y^2-87X^2=169$ |
| 12 | $(10y_{n+2}-541y_{n+1},\ 58y_{n+1}-y_{n+2})$ | $87Y^2-X^2=132327$ |
| 13 | $(10y_{n+3}-30286y_{n+1},\ 3247y_{n+1}-y_{n+3})$ | $87Y^2-X^2=414977472$ |
| 14 | $(541y_{n+3}-30286y_{n+2},\ 3247y_{n+2}-58y_{n+3})$ | $87Y^2-X^2=132327$ |

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**ii)** Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table III below.

Table III: Parabolas

| S. no | (X,Y) | Parabola |
|---|---|---|
| 1 | $\left(58x_{n+1}-x_{n+2},\ 10x_{2n+3}-541x_{2n+2}\right)$ | $174X^2=39Y-1521$ |
| 2 | $\left(3247x_{n+1}-x_{n+3},\ 10x_{2n+4}-30286x_{2n+2}\right)$ | $87X^2=1092Y-2384928$ |
| 3 | $\left(541x_{n+1}-y_{n+2},\ 10y_{2n+3}-5046x_{2n+2}\right)$ | $87X^2=182Y-66248$ |
| 4 | $\left(30286x_{n+1}-y_{n+3},\ 10y_{2n+4}-282489x_{2n+2}\right)$ | $174X^2=20371Y-414977641$ |
| 5 | $\left(3247x_{n+2}-58x_{n+3},\ 541x_{2n+4}-30286x_{2n+3}\right)$ | $64X^2=13Y-507$ |
| 6 | $\left(10x_{n+2}-58y_{n+1},\ 541y_{2n+2}-87x_{2n+3}\right)$ | $87X^2=182Y-66248$ |
| 7 | $\left(541x_{n+2}-58y_{n+2},\ 541y_{2n+3}-5046x_{2n+3}\right)$ | $174X^2=13Y-169$ |
| 8 | $\left(30286x_{n+2}-58y_{n+3},\ 541y_{2n+4}-282489x_{2n+3}\right)$ | $87X^2=182Y-66248$ |
| 9 | $\left(10x_{n+3}-3247y_{n+1},\ 30286y_{2n+2}-87x_{2n+4}\right)$ | $174X^2=20371Y-414977641$ |
| 10 | $\left(541x_{n+3}-3247y_{n+2},\ 30286y_{2n+3}-5046x_{2n+4}\right)$ | $87X^2=182Y-66248$ |
| 11 | $\left(30286x_{n+3}-3247y_{n+3},\ 30286y_{2n+4}-282489x_{2n+4}\right)$ | $174X^2=13Y-169$ |
| 12 | $\left(10y_{n+2}-541y_{n+1},\ 58y_{2n+2}-y_{2n+3}\right)$ | $2X^2=3393Y-132327$ |
| 13 | $\left(10y_{n+3}-30286y_{n+1},\ 3247y_{2n+2}-y_{2n+4}\right)$ | $X^2=95004Y-207488736$ |
| 14 | $\left(541y_{n+3}-30286y_{n+2},\ 3247y_{2n+3}-58y_{2n+4}\right)$ | $X^2=3393Y-132327$ |

**iii)** Consider $m=x_{n+1}, n=x_{n+1}$. Observe that $m>n>0$. Treat m, n as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$, where $\alpha=2mn, \beta=m^2-n^2, \gamma=m^2+n^2$

Then the following interesting relations are observed.

a) $2\alpha-87\beta+85\gamma=-26$

b) $6\gamma-89\alpha+348\frac{A}{P}=26$

c) $\frac{2A}{P}=x_{n+1}y_{n+1}$

## 4. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2=87x^2+13$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine their integer solutions along with suitable properties.

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