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Solution Of Volterra Integro-Differential Equations By Triple Laplace Transform

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ABSTRACT

In this work, we have discussed the different properties of the triple Laplace transform like Linearity property, Convolution theorem property, differential property, triple integral property, and the examples application to solve linear Volterra Integro-differential equations in three-dimensions.

Keyword: *Triple Laplace transform, Inverse triple Laplace transform, Partial integral-differential equations.*

1. INTRODUCTION

Linear integral equations are used to model many original in engineering, chemistry, and physics. Other disciplines, Note that most integral-differential equations give solutions in a closed form. It is therefore important to propose new methods of finding solutions to differential equations [1-3]. Many methods have been developed to solutions the integral-differential equations of two-dimension templates , assembly method, the homotopy perturbation method, the Adomian decomposition method, Laplace transform method, Sumudu transform method, Elzaki transform method and many others. One example of linear integral-differential equations is the three-dimension linear volterra (LVIDEs), which are obtained from modeling in engineering applications. The general three-dimension (LVIDEs) in the following by: [14-15]

$$\frac{\partial^3 u(x,y,t)}{\partial x\,\partial y\,\partial t}+u(x,y,t)=g(x,y,t)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{t_0}^{t}H(x,y,t,k,r,s,u(k,r,s))dkdrds \qquad (1)$$

Where, $u(x,y,t)$ are the unknown function, and the functions H and g are analytic in the domain of interest.

P. S. Laplace (1749–1827) introduced the idea of the Laplace transform in 1782: [2-6]

The Laplace transform denoted by the operator L is defined as

$$L[f(t):\rho)]=\int_0^\infty e^{-\rho t}f(t)dt \;,\quad t>0 \qquad (2)$$

The triple Laplace Transform is denoted by:

$$L_3[f(x,y,t):(\sigma,\rho,\delta)]=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma x-\rho y-\delta t}f(x,y,t)dxdydt$$

Where, $x,y,t>0$ and σ,ρ,δ are Laplace variables (complex values), and

$$f(x,y,t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\sigma x}[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}e^{\rho y}[\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{\delta t}f(\sigma,\rho,t)d\delta]d\rho]d\sigma \qquad (3)$$

is the inverse triple Laplace transform. (denoted by the operator L_3^{-1}).

The structure of this paper is organized as follows: In Section 2, we study the existence and uniqueness of the solution of Eq. (1) , and In Section 3 , we study basic definitions and the use of Linearity property, Convolution theorem property of $f(x,y,t)$, $g(x,y,t)$,definitional and Integral property , and In Section 4 , the examples application to solve linear volterra integro-differential equations in three-dimensions , finally the Conclusions of this paper .

2. ON THE EXISTENCE OF THE SOLUTION (LVIDES) OF THE THREE-DIMENSIONAL [14]

In this subsection, we investigate the existence and uniqueness of solution of Eq. (1) , on the complete metric space of complex valued continuous functions as follows:

$$M=\left[C(S,d)\right], d\left(g,z\right)=\sup\left\{\left|g(x,y,t)-z(x,y,t):(x,y,t)\in S\right|\right\}$$

Where, $S=[0,1]\times[0,1]\times[0,1]$

Theorem: 2.1 Let g and H be continuous functions on $[0,1]^3$ and $[0,1]\times[0,1]\times C$ respectively and there exists nonnegative constant $L\le 1$ such that

$$\begin{aligned}&\left|H(x,y,t,k,r,s,u(k,r,s))-H(x,y,t,k,r,s,v(k,r,s))\right|\\&\qquad\le L\left|u(k,r,s)-v(k,r,s)\right|\end{aligned}$$

Then,

$$u(x,y,t)=g(x,y,t)+\int_0^x\int_0^y\int_0^t H(x,y,t,k,r,s,u(k,r,s))dkdrds$$

has only one continuous solution u on S .

Proof: For the proof, see [14]

Corollary: 2.2 If the hypothesis of theorem 1 hold, then the equation (1)

$$\frac{\partial^3 u(x,y,t)}{\partial x\,\partial y\,\partial t}+u(x,y,t)=g(x,y,t)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{t_0}^{t} H(x,y,t,k,r,s,u(k,r,s))dkdrds$$

with initial condition

$$\begin{aligned}&u(0,0,0)=h_0, u(x,0,0)=h_1(x), u(0,y,0)=h_2(y), u(0,0,t)=h_3(t),\\&u(x,y,0)=h_4(x,y), u(x,0,t)=h_5(x,t), u(0,y,t)=h_6(y,t)\end{aligned}$$

So, by theorem 1, the equation has a unique continuous solution. see [15]

3. THEOREMS AND PROPERTIES OF TRIPLE LAPLACE TRANSFORM

Theorem: 3.1 (linearity of the triple Laplace transform)

Let $f(x,y,t)$ and $g(x,y,t)$ be functions whose the triple Laplace transform exists then

$$L_3\left[\alpha f(x,y,t)+\beta g(x,y,t)\right]=\alpha L_3\left[f(x,y,t)\right]+\beta L_3\left[g(x,y,t)\right]$$

Where, α and β are constants

Theorem: 3.2 (Convolution theorem)

If, $L_3\left[F(x,y,t)\right]=f(\sigma,\rho,\delta)$, $L_3\left[G(x,y,t)\right]=g(\sigma,\rho,\delta)$

and $L_3\left[(F***G)(x,y,t)\right]=\int_0^x\int_0^y\int_0^t F(x-\alpha,y-\beta,t-\kappa)G(\alpha,\beta,\kappa)dxdydt$

then, $L_3\left[(F***G)(x,y,t)\right]=L_3\left[F(x,y,t)\right].L_3\left[G(x,y,t)\right]=f(\sigma,\rho,\delta).g(\sigma,\rho,\delta)$

Property: 3.3

If, $f(x,y,t)=\dfrac{\partial^3 f(x,y,t)}{\partial x\,\partial y\,\partial t}$

Then, $$L_3\left[\frac{\partial^3 f(x,y,t)}{\partial x\,\partial y\,\partial t}:(\sigma,\rho,\delta)\right]=\sigma\rho\delta F(\sigma,\rho,\delta)+\sigma F(\sigma,0,0)+\rho F(0,\rho,0)+$$
$$\delta F(0,0,\delta)-\sigma\rho F(\sigma,\rho,0)-\sigma\delta F(\sigma,0,\delta)-\rho\delta F(0,\rho,\delta)-F(0,0,0)$$

Property: 3.4

If, $L_3\left[f(x,y,t)\right]=\bar{f}(\sigma,\rho,\delta)$ and

$$g(x,y,t)=\int_0^x\int_0^y\int_0^t f(u,v,w)dudvdw$$

Then, $$L_x L_y L_t\left\{\int_0^x\int_0^y\int_0^t f(u,v,w)dudvdw\right\}=\frac{\bar{f}(\sigma,\rho,\delta)}{\sigma\rho\delta}$$

4.APPLICATION OF SOLUTION OF LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS BY TRIPLE LAPLACE TRANSFORM

Example 4.1: Consider the linear volterra integro-differential equation

$$\frac{\partial^3 u(x,y,t)}{\partial x\,\partial y\,\partial t}+u(x,y,t)=x\cos t-\frac{x^2 y\sin t}{2}+\int_0^x\int_0^y\int_0^t u(k,r,s)dkdrds, \qquad \textbf{(4)}$$

With, $u(x,y,0)=x\,,u(x,0,t)=x\cos t,\ u(0,y,t)=0,u(0,0,t)=0$

$$u(x,0,0)=x\,,u(0,y,0)=0,u(0,0,0)=0 \qquad \textbf{(5)}$$

Solution: By taking the triple Laplace transform to Equation (4), and double Laplace transform of initial conditions Equation (5):

$$\sigma\rho\delta\bar{u}(\sigma,\rho,\delta)+\sigma\bar{u}(\sigma,0,0)+\rho\bar{u}(0,\rho,0)+\delta\bar{u}(0,0,\delta)-\sigma\rho\bar{u}(\sigma,\rho,0)-\sigma\delta\bar{u}(\sigma,0,\delta)-$$

$$\rho\delta\bar{u}(0,\rho,\delta)-\bar{u}(0,0,0)+\bar{u}(\sigma,\rho,\delta)=\frac{1}{\sigma^2\rho}\left[\frac{1}{1+\delta^2}\right]-\frac{1}{\sigma^3\rho^2}\left[\frac{\delta}{1+\delta^2}\right]+\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta) \qquad \textbf{(6)}$$

$$\bar{u}(0,\rho,\delta)=0\ ,\ \bar{u}(\sigma,0,\delta)=\frac{1}{\sigma^2}\left[\frac{\delta}{\delta^2+1}\right],\bar{u}(\sigma,\rho,0)=\frac{1}{\sigma^2\rho},\bar{u}(0,0,\delta)=0,\bar{u}(0,\rho,0)=0,$$

$$\bar{u}(\sigma,0,0)=\frac{1}{\sigma^2},\bar{u}(0,0,0)=0 \qquad \textbf{(7)}$$

Substituting (7) in (6), simplifying, and we obtain,

$$\bar{u}(\sigma,\rho,\delta)\left[\sigma\rho\delta+1-\frac{1}{\sigma\rho\delta}\right]=\frac{1}{\sigma^2\rho}\left[\frac{\delta}{1+\delta^2}\right]-\frac{1}{\sigma^3\rho^2}\left[\frac{1}{1+\delta^2}\right]+\frac{1}{\sigma}\left[\frac{\delta^2}{\delta^2+1}\right]$$

$$\bar{u}(\sigma,\rho,\delta)\left[\sigma\rho\delta+1-\frac{1}{\sigma\rho\delta}\right]=\frac{1}{\sigma^2\rho}\left[\frac{\delta}{1+\delta^2}\right]\left[1-\frac{1}{\sigma\rho\delta}+\sigma\rho\delta\right]$$

$$\bar{u}(\sigma,\rho,\delta)=\frac{1}{\sigma^2\rho}\left[\frac{\delta}{\delta^2+1}\right]$$

Appling inverse triple Laplace transform both sides, $u(x,y,t)=x\cos t$

Example 4.2: Consider the linear volterra integro-differential equation

$$\frac{\partial^3 u(x,y,t)}{\partial x\,\partial y\,\partial t}+u(x,y,t)=xyt+1-\frac{x^2y^2t^2}{8}+\int_0^x\int_0^y\int_0^t u(k,r,s)dkdrds, \qquad \textbf{(8)}$$

With $u(x,y,0)=u(x,0,t)=u(0,y,t)=0$ **(9)**

Solution: By taking the triple Laplace transform to Equation (8), and double Laplace transform of initial conditions Equation (9):

$$\sigma\rho\delta\bar{u}(\sigma,\rho,\delta)+\sigma\bar{u}(\sigma,0,0)+\rho\bar{u}(0,\rho,0)+\delta\bar{u}(0,0,\delta)-\sigma\rho\bar{u}(\sigma,\rho,0)-$$

$$\sigma\delta\bar{u}(\sigma,0,\delta)-\rho\delta\bar{u}(0,\rho,\delta)-\bar{u}(0,0,0)+\bar{u}(\sigma,\rho,\delta)=\frac{1}{\sigma^2\rho^2\delta^2}+\frac{1}{\sigma\rho\delta}-\frac{1}{\sigma^3\rho^3\delta^3}+\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta) \qquad \textbf{(10)}$$

$$\bar{u}(\sigma,\rho,0)=0\quad,\bar{u}(\sigma,0,\delta)=0\ ,\bar{u}(0,\rho,\delta)=0 \qquad \textbf{(11)}$$

Substituting (11) in (10), simplifying, and we obtain,

$$\bar{u}(\sigma,\rho,\delta)\left[\sigma\rho\delta+1-\frac{1}{\sigma\rho\delta}\right]=\frac{1}{\sigma^2\rho^2\delta^2}+\frac{1}{\sigma\rho\delta}-\frac{1}{\sigma^3\rho^3\delta^3}$$

$$\bar{u}(\sigma,\rho,\delta)\left[\frac{1}{\sigma\rho\delta}+1-\sigma\rho\delta\right]=\frac{1}{\sigma^2\rho^2\delta^2}\left[1+\sigma\rho\delta-\frac{1}{\sigma\rho\delta}\right]$$

$$\bar{u}(\sigma,\rho,\delta)=\frac{1}{\sigma^2\rho^2\delta^2}$$

Appling inverse triple Laplace transform both sides,

$$u(x,y,t)=xyt$$

Example 4.3: Consider the linear volterra integro-differential equation

$$\frac{\partial^3 u(x,y,t)}{\partial x\,\partial y\,\partial t}+u(x,y,t)=\frac{x^2yt+xy^2t+xyt^2}{2}+x+y+t-\int_0^x\int_0^y\int_0^t u(k,r,s)dkdrds \quad \textbf{(12)}$$

With $u(x,y,0)=x+y$, $u(x,0,t)=x+t$, $u(0,y,t)=y+t$, $u(0,0,t)=t$

$$u(x,0,0)=x,\ u(0,y,0)=y,\ u(0,0,0)=0 \quad \textbf{(13)}$$

Solution: By taking the triple Laplace transform to Eq. (12), and double Laplace transform of initial conditions Eq.(13)

$$L_3\left[\frac{\partial^3 u}{\partial x\,\partial y\,\partial t}\right]+L_3\left[u(x,y,t)\right]=L_3\left[\frac{x^2yt+xy^2t+xyt^2}{2}+x+y+t-\int_0^x\int_0^y\int_0^t u(k,r,s)dkdrds\right]$$

$$\sigma\rho\delta\bar{u}(\sigma,\rho,\delta)+\sigma\bar{u}(\sigma,0,0)+\rho\bar{u}(0,\rho,0)+\delta\bar{u}(0,0,\delta)-\sigma\rho\bar{u}(\sigma,\rho,0)-$$
$$\sigma\delta\bar{u}(\sigma,0,\delta)-\rho\delta\bar{u}(0,\rho,\delta)-\bar{u}(0,0,0)+\bar{u}(\sigma,\rho,\delta)=\frac{1}{\sigma^3\rho\delta}+\frac{1}{\sigma\rho^3\delta}+\frac{1}{\sigma\rho\delta^3}+$$
$$\frac{1}{\sigma^2\rho\delta}+\frac{1}{\sigma\rho^2\delta}+\frac{1}{\sigma\rho\delta^2}-\frac{1}{\sigma\rho\delta}\bar{u}(\sigma,\rho,\delta)$$

$$\bar{u}(\sigma,\rho,0)=\left[\frac{1}{\sigma^2\rho}+\frac{1}{\sigma\rho^2}\right],\ \bar{u}(\sigma,0,\delta)=\left[\frac{1}{\sigma^2\delta}+\frac{1}{\sigma\delta^2}\right],\ \bar{u}(0,\rho,\delta)=\left[\frac{1}{\rho^2\delta}+\frac{1}{\rho\delta^2}\right]$$

$$\bar{u}(\sigma,0,0)=\frac{1}{\sigma^2},\ \bar{u}(0,\rho,0)=\frac{1}{\rho^2},\ \bar{u}(0,0,\delta)=\frac{1}{\delta^2},\ \bar{u}(0,0,0)=0$$

$$\bar{u}(\sigma,\rho,\delta)\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]=\frac{1}{\sigma^2\rho\delta}\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]+\frac{1}{\sigma\rho^2\delta}\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]+$$
$$\frac{1}{\sigma\rho\delta^2}\left[\frac{1}{\sigma\rho\delta}+\sigma\rho\delta+1\right]$$

$$\bar{u}(\sigma,\rho,\delta)=\frac{1}{\sigma^2\rho\delta}+\frac{1}{\sigma\rho^2\delta}+\frac{1}{\sigma\rho\delta^2}$$

Appling inverse triple Laplace transform both sides,

$$u(x,y,t)=x+y+t$$

5. CONCLUSION

This work discussed the definition of the triple Laplace transform that was applied Some important theorems and properties have been presented for this relatively new transformation to find solutions for linear volterra integro-differential equations in three-dimensions. under the initial conditions, the triple Laplace transform study succeeded in achieving solutions.

REFERENCES

[1] Abdul Majid Wazwaz, Partial Differential Equations and Solitary Waves Theory, Higher Education Press Beijing and Springer - Verlag Berlin Heidelberg (2009).

[2] Moghadam MM, Saeedi H Application of differential transforms for solving the Volterra integro-partial differential equations. Iranian Journal of Science and Technology, Transaction A 34(A1):59 (2010

[3] Hassan Eltayeb., Adem Kilicman, On double Sumudu transform and double Laplace transform, Malaysian journal of Mathematical Sciences, 2010.

[4] Tarig M. Elzaki , Adil Mousa On the convergence of triple Elzaki transform SN Applied Sciences 1:275 | https://doi.org/10.1007/s42452-019-0257-2 (2019)

[5] Hassan Eltayeb., Adem Kilicman, A note on double Laplace transform, and Telegraphic equation, Hindawi Publishing Corporation, 2013.

[6] Pmar Özel.,LÖzlem Bayar, The double Laplace transform , January 2012.

[7] Prem Kiran G., Bhadane. V. H. Pradhan, and Satish V. Desale, Elzaki transform solution of a one dimensional groundwater Recharge through spreading, P. G. Bhandane et al int. Journal of Engineering Research and applicat ions, ISSN 2248-9622,Vol. 3, Issue 6, (2013).

[8] Abdon Atangana A Note on the Triple Laplace Transform and Its Applications to Some Kind of Third-Order Differential Equation Abstract and Applied Analysis, Article ID 769102, 10 pages Volume (2013)

[9] Trig M. Elzaki, Salih M. Elzaki, Application of new transform " Elzaki Transform" to partial differential equation, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768, Number1 (2011), pp. 65-70.

[10] Trig M. Elzaki, Salih M. Elzaki, On connection between Laplace transform and Elzaki Transform , Advances in Theoretical and Applied Mathematics, ISSN0973-4554 volume 6 Number1(211), pp. 1-11.

[11] Trig M. Elzaki, Salih M. Elzaki, and Elsayed A. Elnour, On the new Integral transform "Elzaki Transform" fundamental properties investigation and application, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768, Number 4number1(2012), pp. 1-13.

[12] Trig M. Elzaki, The New Integral transform " Elzaki Transform" Global Journal of Pure and Applied Mathematics, ISSN 0973- 1768, Number1, (2011), pp. 57-64.

[13] Trig M. Elzaki, Solution of Nonlinear Differential Equations Using Mixture of Elzaki Transform and Differential transform Method, International Mathematical Forum, vol.7, (2012), no. 13, 631-638.

[14] Farshid Mirzaee , Elham Hadadiyan, Three-dimensional triangular functions and their applications for solving nonlinear mixed Volterra–Fredholm integral equations Alexandria Engineering Journal (2016) 55, 2943–2952

[15] Jalil Manafian, Peyman Bolghar, Numerical solutions of nonlinear 3-dimensional Volterra integral-differential equations with 3D-block-pulse functions, wileyonlinelibrary.com/journal/mma, DOI: 10.1002/mma.4936